# My Collaboration with Narasimhan– A short account

S.Ramanan

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October 2021

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Somehow Fate conspired in my favour to have him as my mentor. Our first work together was on

Universal Connections.

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Soon after MSN returned, he was happy to find that I was familiar with the work of Kodaira and Spencer and organized a seminar on the topic, jointly with me and with a colleague of ours, called Simha with whom he later wrote an interesting paper.

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I shall not dwell on it any more!

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We tried to check that these are the only smooth points and succeeded only when the genus $g \geq 3$ and in the case of $g = 2$, when the rank of the VBs in question is $\geq 3$. When $g = 2$ and $n = 2$, the moduli space $M^0$ of VBs with trivial determinant is a 3-dimensional normal variety and the non-stable points constitute a surface. So this is not true in that case. We found it intriguing.

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Soon, we proved that in this special case, the whole variety was smooth!

**I may mention here that at that time our knowledge of algebraic curves - we thought of them mostly as Riemann surfaces – was rudimentary. For example, we guessed and proved for ourselves that all curves of genus 2 are hyper-elliptic and that the hyper-elliptic involution was unique!**

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But quite soon, we managed to show that the moduli space $M^0$ of VBs with trivial determinant was actually the 3-dimensional projective space.

We proved this initially by giving some criterion for a 3-dimensional variety to be a projective space, and checking it for the moduli variety. Our colleague C.P. Ramanujam took this point of view seriously and came up with nice characterisations of projective spaces in general.

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Since the Grassmannian of lines in $P^3$ is a quadric in $P^5$, it was a question of determining the divisor $M^1$ in this quadric. We were not aware then of the relationship of the pencil of quadrics with the hyper-elliptic curve. It took a long time for us to determine it and when we did, we were delighted to learn that this geometry went back more than a century!

**Just as one can associate a line in $M^0$ to a VB in $M^1$, we realised that to every VB of $M^0$ (at least when it is stable) one could associate a projective line in $M^1$. Going back and forth, i.e. starting with a VB in $M^0$ one can associate this line in $M^1$ and hence get a quadratic cone back in $M^0$.**

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**Since I had heard of the idea of quadratic complexes, we started looking in projective geometriy books for a good understanding of this corrspondence. We finally found that there is a particular quadric in this $P^5$ associated to the curve and we guessed and showed that $M^1$ was the intersection of the Grassmannian and that quadric.**

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A great help in this regard was a paper of Klein which had been published in Vol. 2 of Mathematische Annalen! Every geometric idea in this paper has a corresponding interpretation in terms of the VB moduli. The exhaustive Library at TIFR was a great help to us indeed!

In writing out the details, we needed the fact (which we could verify) that that there is a Poincare bundle for $M^1$. Seshadri was at that time visiting Harvard and I wrote to him, among other things, that there is a Poincare bundle in $M^1$ and that we managed to prove this was true whenever the rank and degree were co-prime.

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Fortunately, we had in the meanwhile proved that there is no Poincare bundle in the degree 0 case in the case of rank 2, genus 2, thanks to our geometric understanding of this special case!

I could later show that the degree and rank being relatively prime is a necessary and sufficient condition for the existence of a universal bundle.

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We wondered if the analogue of Torelli's theorem was valid for these moduli varieties as well. Not only was this true, but the significant difference in this regard is that unlike the Jacobian which of course can be deformed into abelian varieties which are not necessarily Jacobians, the deformations of the moduli varieties of fixed determinant are obtained only from deformations of the curve.

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**We could also determine the group of automorphisms of the $M^1$ in terms of that of the curve.**

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As far as explicit determination of the moduli varieties is concerned, we had initially considered in detail only the case of curves of genus 2. Later we began studying the case of genus 3 and in this case, an earlier purely geometric study by Coble was very helpful.

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The Moduli space in the case of rank 2 and trivial determinant behaves in a different way depending on whether the curve is hyper-elliptic or not. When it is not hyper-elliptic, it is imbedded as a quartic hyper-surface in the projective space of the linear system $2\Theta$ on the Jacobian. When it is hyper-elliptic, the hyper-elliptic involution acts on $M^0$ and the quotient is a quadric.

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Our geometric understanding of the Quadratic Complex theory and the explicit determination of the moduli spaces when the genus is 2, gave us a felicitous tool in general.

This is the correspondence between Moduli with determinant $\xi$ and Moduli with determinant $\xi \otimes \mathcal{O}(x)$. We called this *Hecke correspondence*.

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On his way back to India, MSN stopped in France and met Arnaud. He found that Beauville had also discovered the same usefulness of passing to a ramified covering and taking direct images in this context.

We decided to write this up jointly rather than publish two articles independently. This paper led to developments in many directions, for example to a study of principal $G$-bundle pairs as well, by Donagi and others.

**Apart from the theory of VBAC on the contributions to which I have been focussing so far, we also used to discuss other questions from time to time. One of them is our understanding of the Weizenbock formula. This expresses the Laplacian of a Riemannian manifold, in terms of its symbol and other constructions.**

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The significance of this formula is that various theorems regarding cohomology can be derived from this, using the identication of the cohomology space with harmonic forms.

MSN and I understood this in a general set-up and we could set it up in terms of symbol-lifting property of connections and could explain many vanishing theorems such as Bochner's theorem, and Kodaira's vanishing theorem, etc. I later worked it up in detail and included it in my book, 'Global Calculus'.

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My first student – an excellent one indeed - A.Ramanathan received a lot of help in his early work from MSN. He and Shrawan Kumar, a later student of mine, collaborated with MSN and wrote up an excellent paper.

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On the other side, I had the opportunity to be of help to a student of his – a brilliant one - Patodi.

**Finally, I would like to mention our collaboration in Administrative activity. In the early 80's, I was serving as the Dean of the School of Mathematics. At that time there was talk of starting a Mathematics Council, which would be the principal Funding Agency for Mathematics Programmes.**

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The Department of Atomic Energy which was in charge of Higher Mathematics, decided to constitute the 'National Board for Mathematics'. The Chairman of DAE consulted me and requested me to suggest a name for Chairmanship. I had no doubt that MSN fitted the Bill.

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Accordingly, MSN as chairman, and I, as Secretary of the Board, functioned together in setting it up in the initial years.

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- **He wrote an article on our collaboration, which was published as part of the proceedings of that conference.**

# The Work of S. Ramanan

M.S. Narasimhan

ABSTRACT. This talk gives an overview of some of the significant work of S. Ramanan in the areas of moduli problems, linear systems on flag varieties and abelian varieties, and differential geometry.

## 1. Introduction

It is with great pleasure that I write this account of the work of S.Ramanan. I have been fortunate enough to have collaborated with him intensely over a long period. I have profited much from his mathematical insights.

His mathematics is characterised by depth and breadth and covers several areas: algebra, differential geometry and algebraic geometry. In algebraic geometry he is a major figure, and his main interests have been:

- moduli of vector bundles on curves,
- homogeneous vector bundles on flag varieties,
- abelian varieties,
- geometric invariant theory,
- linear systems on flag varieties in positive characteristics,
- Green's conjecture on syzygies of canonical curves, and
- Higgs bundles.

His contributions to the study of vector bundles on curves have been particularly extensive and profound.

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