Mixed Hodge Structures of Character Varieties of Free Groups

Azizeh Nozad

School of Mathematics, IPM, Tehran, Iran

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Outline

Character varieties

- G complex reductive affine algebraic group $(G = GL(n,\mathbb{C}), PGL(n,\mathbb{C}), SL(n,\mathbb{C}), Sp(n,\mathbb{C})\ldots)$.
- Γ finitely presented group, $\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_r | r_1, r_2, \ldots r_s \rangle$
- $\mathscr{R}_\Gamma G := \mathrm{Hom}(\Gamma, G)$ G-representation variety of Γ

G-Character variety of Γ

$\mathscr{X}_\Gamma G := \mathscr{R}_\Gamma G // G$, GIT quotient, under conjugation: $\rho \in \mathrm{Hom}(\Gamma, G)$, $(g \cdot \rho)(\gamma) := g\rho(\gamma)g^{-1}$, $g \in G$, $\gamma \in \Gamma$.

Example

- *with* $\Gamma = F_r$ *free group of rank* r *we have* $\mathscr{X}_r G := \mathscr{X}_{F_r} \cong G^r // G$
- *with* $\Gamma = \mathbb{Z}^r$ *free abelian group we have* $\mathscr{R}_{\mathbb{Z}^r} G = \{(g_1, \cdots, g_r) : \ g_i g_j = g_j g_i\} \subset G^r$ *Commuting variety of* r*-tuples in* G *and* $\mathscr{X}_{\mathbb{Z}^r} = \mathscr{R}_{\mathbb{Z}^r}(G) // G$

Motivation

- (Topology/Differential Geometry) Space of flat G-connections on a manifold M with $\pi_1(M) = \Gamma$:

There is a natural *one-to-one correspondence*:
$$\hom(\Gamma, G)//G = \mathscr{X}_\Gamma G \quad \leftrightarrows \quad \{\text{flat connections on } P\}/\text{Gauge}$$

- (Algebra) Matrix invariants under simultaneous conjugation.
- (Knot theory) A-polynomial defined by the image of a morphism between character varieties: $\mathscr{X}_\Gamma SL_2(\mathbb{C}) \to \mathscr{X}_{\mathbb{Z}_2} SL_2(\mathbb{C})$.
- Non-abelian Hodge correspondence:

Theorem (Hitchin, Donaldson, Corlette, Simpson 1986-90)

Let $\Gamma = \pi_1(M)$, M a Riemann surface and G be a real or complex reductive Lie group. Then the character variety $\mathscr{X}_\Gamma G = \hom(\Gamma, G)//G$ is homeomorphic to $\mathscr{H}_M G$, a moduli space of G-Higgs bundles over M.

- (Mirror symmetry) Equality of invariants for Langlands dual G and $^L G$-structures.

Mixed Hodge structures on cohomology by Deligne

Deligne has extended Hodge theory (Hodge decomposition of the de Rham cohomology of Kähler variety) to algebraic varieties which may be singular or noncompact.

Mixed Hodge Structures

- X quasi-projective algebraic variety $/\mathbb{C}$.
- Singular cohomology with compact support $H_c^k(X)$.
- Deligne defines a natural and functorial mixed Hodge structure $H_c^{k,p,q} = H^{p,q}[H_c^k(X,\mathbb{C})]$.
- Mixed Hodge numbers $h^{k,p,q} := \dim_{\mathbb{C}} H_c^{k,p,q}$.
- Compactly supported Betti numbers $\dim H_c^k(X) = \sum_{p,q} h^{k,p,q}(X)$ and (p,q) with $h^{k,p,q}(X) \neq 0$ are the k-weights of $H_c^k(X) \Rightarrow$ Give usual Betti numbers by Poincaré duality in the smooth case.

- $h^{k,p,q}(X) = h^{k,q,p}(X)$.
- Kähler varieties carry pure Hodge structure: only weights $(p, k-p)$.
- Hodge-Tate varieties, or balanced, only weights (p,p).

Polynomials

We can assemble all the $h^{k,p,q}(X)$ in the (compactly supported) *mixed Hodge polynomial*

$$\mu(X;t,u,v) := \sum_{k,p,q\geq 0} h^{k,p,q}(X)\ t^k u^p v^q \in \mathbb{N}_0[t,u,v],$$

Then: (compactly supported) Poincaré polynomial $P(X;t) = \mu(X;t,1,1)$ and the Serre (E-) polynomial is $E(X;u,v) = \mu(X;-1,u,v)$.
Also, compactly supported Euler characteristic $\chi^c(X) = E(X;\ 1,1) = \mu(X;1,1,1) = P^c(X;\ -1) = \sum_k (-1)^k \dim H_c^k(X)$ coincides with $\chi(X)$ for quasi-projective.

Multiplicativity (Künneth) and Additivity

$E(X \times Y) = E(X) \cdot E(Y)$
$X = Z \sqcup (X \backslash Z)$ locally closed $\Rightarrow E(X) = E(Z) + E(X \backslash Z)$

Example

$E(\mathbb{C}^n) = (uv)^n$, $E((\mathbb{C}^*)^n) = (uv-1)^n$, so that $\chi(\mathbb{C}^n) = 1$, $\chi((\mathbb{C}^*)^n) = 0$.

E-polynomial under fibrations

Proposition (Dimca-Lehrer, Logares-Newstead-Muñoz)

Let $F \to^{W \curvearrowright} X \to B$, W preserving fibers $\pi^{-1}(b)$ verifying any of

a) *Locally Zariski trivial (LTZ)*
b) *Smooth, locally analytic trivial and $\pi_1(B) \curvearrowright H_c^*(F)$ trivially*
c) *X, B smooth and F complex connected Lie group*
d) *F is a special group (all principal F-bundles are LZT)*

$$\text{then,}\quad E^W(X) = E^W(F) \cdot E(B)\,.$$

Of course, if W is trivial, $E(X) = E(F) \cdot E(B)$.

- From $\mathbb{C}^* \to GL_n \to PGL_n$, $E(GL_n) = (1 - uv) \cdot E(PGL_n)$.

Example (F needs to be connected)

$$\mathbb{Z}_2 \to \mathbb{P}^1 \times \mathbb{P}^1 \to \mathrm{Sym}^2(\mathbb{P}^1), \quad (1 + uv)^2 \neq 2 \cdot (1 + uv + u^2 v^2)$$

Some results

Let $\Gamma = \pi_1(\Sigma_g)$ the fund. group of genus g compact orientable surface.

- N. Hitchin ('87): Poincaré polynomials for $G = SL_2\mathbb{C}$
- P. Gothen ('94): Poincaré polynomials for $G = SL_3\mathbb{C}$
- T. Hausel - F. Rodriguez-Villegas (2008): Hodge-Deligne polynomials for $SL_2\mathbb{C}$, conjectures for higher n.
- Logares, Muñoz, Newstead, Martínez ('13,'14,'17): Geometric approach to compute E-polynomials for $G = SL_2(\mathbb{C})$ and $PGL_2(\mathbb{C})$
- Mozgovoy-Reineke: ('15) Generating series to compute the points of $\mathscr{X}_r GL_n$ over $\mathbb{F}_q$, in terms of irreducible representations.
- Garcia-Prada, Heinloth and Schmitt ('14); Schiffman ('16), Mellit ('17): Poincaré polynomials for all $G = SL_n\mathbb{C}$.
- Florentino-Silva ('18): Combined methods for abelian character varieties $\mathscr{X}_{\mathbb{Z}^r} G$, $G = GL_n(\mathbb{C}), SL_n(\mathbb{C}), Sp_n(\mathbb{C})$.

Most of the above results are for smooth (twisted) character varieties. Not much is known for $Hom(\Gamma, G)//G$ except for low rank SL_n or GL_n.

Free group case: Deformation Retraction (Homotopy equivalences)

Let $\Gamma = F_r$ the free group of rank r. Use the notation $\mathscr{X}_r G := \mathscr{X}_{F_r} G$, recall: $\mathscr{X}_r G = G^r /\!/ G$. Topology result in the free case $\Gamma = F_r$:

Theorem (Florentino-Lawton-Casimiro-Oliveira '09-'15)

Let G be a real/complex reductive group, with maximal compact subgroup K. Then, $\mathscr{X}_{F_r} K$ is a strong deformation retraction of $\mathscr{X}_{F_r} G$ (hence Betti numbers agree $b_k(\mathscr{X}_{F_r} K) = b_k(\mathscr{X}_{F_r} G)$ for all k).

Example: Tom Baird's formula, for $SU(2)$, implies the one for SL_2

$$P(\mathscr{X}_r SL_2\mathbb{C};\, t) = 1 + t - \frac{t(1+t^3)^r}{1-t^4} + \frac{t^3}{2}\left(\frac{(1+t)^r}{1-t^2} - \frac{(1-t)^r}{1+t^3}\right)$$

Relation with number theory

Let X be a $\mathbb{Z}$-scheme and let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements.

Polynomial count

We say X is polynomial count if there is a *counting polynomial* for X, $CX(t) \in \mathbb{Z}[t]$ such that $|X/\mathbb{F}_q| = CX(q)$, for almost every prime p.

Theorem (Katz ('08))

If X is polynomial count then $E(X_{\mathbb{C}}; u, v) = CX(uv)$, where $X_{\mathbb{C}} := X \otimes_{\mathbb{Z}} \mathbb{C}$.

Theorem (Mozgovoy-Reineke ('15))

When $\Gamma = F_r$, free group of rank r, (full and irreducible) GL_n-character varieties are polynomial count.

Arithmetic: counting points over finite fields

Mozgovoy-Reineke '15:

$$\sum_{n\geq 1} E(\mathscr{X}^{irr}_{F_r} GL_n)\, t^n = (1-x) PLog(S \circ F^{-1}(t))$$

$$\sum_{n\geq 0} E(\mathscr{X}_{F_r} GL_n)\, t^n = PExp\left(\sum_{n\geq 1} E(\mathscr{X}^{irr}_{F_r} GL_n)\, t^n\right),$$

where F and S are the operators in $\mathbb{Q}[x][[t]]$:

$$S(t) = t, S(t^n) := x^{(1-r)\binom{n}{2}} t^n, \qquad F(t) := 1 + \sum_{n\geq 1} ((x-1)\cdots(x^n-1))^{r-1} t^n.$$

Plethystic operators

Adams operator: Given $f(x,y,z) = \sum_n f_n(x,y) z^n \in \mathbb{Q}[x,y][[z]]$, $\Psi(x^i t^k) = \sum_{l\geq 1} \frac{x^{li} t^{lk}}{l}$ and $\Psi^{-1}(x^i t^k) = \sum_{l\geq 1} \frac{\mu(l)}{l} x^{li} t^{lk}$, where μ is the Möbius function $\mu : \mathbb{N} \to \{0, \pm 1\}$ ($\mu(n) = (-1)^k$ if n is square free with k primes in its factorization; $\mu(n) = 0$ otherwise).

$$PExp(f) = e^{\Psi(f)}, \quad PLog(f) := \Psi^{-1}(log f)$$

Stratification by stabilizer dimension

Any character variety admits a stratification by the dimension of the stabilizer of a given representation.

- Let Γ a finitely presented group and G a complex reductive algebraic group
- Locally closed stratification by stabilizer dimension:

$$\mathscr{X}_\Gamma G = \bigsqcup_{m \geq m_0} \mathscr{X}_\Gamma^m G$$

where $m_0 = dim(\bigcap_{\rho \in \mathscr{R}_\Gamma G} \mathrm{Stab}(\rho))$, center of the action of G on $\mathscr{R}_\Gamma G$.

In the linear case $G = GL_n$ (as well as the related groups SL_n and PGL_n), there is a more convenient refined stratification that gives a lot of information on the corresponding character varieties which we call stratification by partition type.

stratification by partition type for $G = GL_n$

Partition

$[k] = [1^{k_1} 2^{k_2} \cdots n^{k_n}] \in \mathscr{P}_n$, $\sum_{j=1}^{n} j \cdot k_j = n$, with length $|[k]| = \sum_{j=1}^{n} k_j$. For example $[1^2\,2\,4] \in \mathscr{P}_8$, whose length is 4.

$[k]$-strata

$\rho \in \mathscr{R}_\Gamma(GL_n)$ is $[k]$-polystable if $\rho \sim_{conj} \bigoplus_{j=1}^{n} \rho_j$, where $\rho_j \in \mathscr{R}_\Gamma(GL_j^{irr})^{\oplus k_j}$. Define $\mathscr{X}_\Gamma^{[k]} GL_n := \mathscr{R}_\Gamma^{[k]} GL_n // GL_n$, the $[k]$-stratum of $\mathscr{X}_\Gamma GL_n$.

- Abelian stratum: $\mathscr{X}_\Gamma^{[1^n]} GL_n$ (of maximal length)
- Irreducible stratum: $\mathscr{X}_\Gamma^{[n]} GL_n$ (of minimal length), smooth locus for GL_n.

Stratification by polystability type for $G = GL_n$

Theorem (Florentino-N. -Zamora (2021))

There exists a locally closed stratification by partition type:

$$\mathscr{X}_\Gamma GL_n = \bigsqcup_{[k]\in\mathscr{P}_n} \mathscr{X}_\Gamma^{[k]} GL_n$$

This stratification refines the one by stablizer dimension and different partitions with the same length become disjoint irreducible components of each stratum by stablizer dimension.

Setting: Γ a finitely presented group. $G = GL_n(\mathbb{C})$. Stratify $\mathscr{X}_\Gamma GL_n$ by *block type*, each strata $\mathscr{X}_\Gamma^{[k]} GL_n$ corresponds to a partition $[k] \in \mathscr{P}_n$. The next result relates, by the plethystic exponential, the generating functions of the E-polynomials $E(\mathscr{X}_\Gamma GL_n)$ to the corresponding generating functions of the E-polynomials of the irreducible character varieties $E(\mathscr{X}_\Gamma^{irr} GL_n)$.

Theorem (Florentino–N.–Zamora (2021))

Let Γ be a finitely presented group. Then, in $\mathbb{Q}[u,v][[t]]$:

$$\sum_{n\geq 0} E(\mathscr{X}_\Gamma GL_n; u, v)\, t^n = PExp\left(\sum_{n\geq 1} E(\mathscr{X}_\Gamma^{irr} GL_n; u, v)\, t^n\right).$$

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Generalizes [Mozgovoy-Reineke '15] to an arbitrary Γ even if the character variety is not Polynomial Type!.

Symmetric Products of Irreducible Character Varieties

$[k]$-Levy

$L_{[k]} := GL_1^{k_1} \times GL_2^{k_2} \times \cdots \times GL_n^{k_n} \subset GL_n$

$L_{[k]}$ acts naturally, factorwise, on the space of polystable representations of type $[k]$, $\mathscr{R}_\Gamma^{[k]} G$, and the GIT quotient is a product of irreducible character varieties:

- $\mathscr{R}_\Gamma^{[k]} GL_n // L_{[k]} = \times_{j=1}^{n} (\mathscr{X}_\Gamma^{irr} GL_j)^{\times k_j}$

Note, however, that this does not coincide with the $[k]$-character variety $\mathscr{X}_\Gamma^{[k]} GL_n$. Indeed, when some $k_j > 1$, there is a permutation group acting on $\mathscr{R}_\Gamma^{[k]} G$ by permuting the blocks of equal size:

$[k]$-symmetric group

$S_{[k]} := S_{k_1} \times S_{k_2} \times \cdots \times S_{k_n} \subset S_n$

each subgroup $S_{k_j} \subset S_n$ only permutes the k_j blocks of size j, and does not act on other blocks. Hence,

Proposition (Florentino- N. -Zamora (2021))

a) $\mathscr{X}_\Gamma^{[k]} GL_n \simeq (\mathscr{R}_\Gamma^{[k]} GL_n // L_{[k]}) / S_{[k]} \simeq \times_{j=1}^{n} \mathrm{Sym}^{k_j} (\mathscr{X}_\Gamma^{irr} GL_n)$.

b) $\sum_{n \geq 0} E(\mathrm{Sym}^n(X); u, v) y^n = \mathrm{PExp}(E(X; u, v) y)$ *(using Cheah).*

Our proofs allow us to obtain a closed formula for each individual E-polynomial of $\mathscr{X}_\Gamma GL_n$ as a finite sum in the E-polynomials of the irreducible character varieties $\mathscr{X}_\Gamma^{irr} GL_n$ of lower dimension: indexed by what we call *rectangular partitions of* n.

Corollary

Let Γ *be a finitely presented group. Then,*

$$E(\mathscr{X}_\Gamma GL_n; u, v) = \sum_{[[k]]\in\mathscr{R}\mathscr{P}_n} \prod_{l,h=1}^{n} \frac{B_l^\Gamma(u^h, v^h)^{k_{l,h}}}{k_{l,h}!\, h^{k_{l,h}}}$$

Moreover, for a given $[m] \in \mathscr{P}_n$*, the* E*-polynomial of the corresponding stratum is:*

$$E(\mathscr{X}_\Gamma^{[m]} GL_n; u, v) = \sum_{[[k]]\in\pi^{-1}[m]} \prod_{l,h=1}^{n} \frac{B_l^\Gamma(u^h, v^h)^{k_{l,h}}}{k_{l,h}!\, h^{k_{l,h}}}$$

where $B_l^\Gamma(u, v) := E(\mathscr{X}_\Gamma^{irr} GL_l; u, v)$.

Rectangular partitions

Rectangular partition

$[[k]] = [(1\times 1)^{k_{1,1}}(1\times 2)^{k_{1,2}}\cdots(1\times n)^{k_{1,n}}\cdots(n\times n)^{k_{n,n}}] \in \mathscr{R}\mathscr{P}_n$ satisfying $n = \sum_{l,h=1}^{n} l\, h\, k_{l,h}$.

The geometric interpretation of rectangular partitions is as follows: we are decomposing an initial set with area n, into a set of rectangles of each possible size $l\times h \leq n$, and each $l\times h$ rectangle appears with multiplicity $k_{l,h}$. This explains the terminology *gluing map* as it is obtained by gluing all rectangles to form the usual Young diagram of a partition:

Gluing map

$$\begin{aligned}\pi : \mathscr{R}\mathscr{P}_n &\to \mathscr{P}_n\\ [[k]] &\mapsto [m] = [1^{m_1}\cdots n^{m_n}]\end{aligned}$$

defined by $m_l := \sum_{h=1}^{n} h\cdot k_{l,h}$

Explicit computations in the free group case

Recall by [Mozgovoy-Reineke ('15), Katz ('08)], for $\Gamma = F_r$, GL_n-character varieties are polynomial count and $E(X;u,v) = CX(uv)$.

Proposition (Mozgovoy-Reineke ('15), Florentino-N. -Zamora ('21))

For $r, n \geq 2$, we have $E(\mathscr{X}_r^{irr} GL_n; x) =$

$$(x-1)\sum_{d|n}\frac{\mu(n/d)}{n/d}\sum_{[k]\in\mathscr{P}_d}\frac{(-1)^{|[k]|}}{|[k]|}\binom{|[k]|}{k_1,\cdots,k_d}\prod_{j=1}^{d} b_j(x^{n/d})^{k_j} x^{\frac{n(r-1)k_j}{d}\binom{j}{2}},$$

where μ is the Möbius function, and the $b_j(x)$ are polynomials defined by

$$\left(1+\sum_{n\geq 1} b_n(x)\, t^n\right)\left(1+\sum_{n\geq 1}\left((x-1)(x^2-1)\dots(x^n-1)\right)^{r-1} t^n\right) = 1\,.$$

Explicit expressions for $B_n^r(x) = E(\mathscr{X}_r^{irr}GL_n; x)$

Example ($s = r-1$)

$$\begin{aligned}
\frac{B_1^r(x)}{x-1} =&(x-1)^s,\\
\frac{B_2^r(x)}{x-1} =&(x-1)^s\Big((x-1)^s x^s((x+1)^s-1)+\frac{1}{2}(x-1)^s-\frac{1}{2}(x+1)^s\Big),\\
\frac{B_3^r(x)}{x-1} =&(x-1)^s\Big(-\frac{1}{3}(x^2+x+1)^s+(x-1)^{2s}(\frac{1}{3}-x^s+x^s(x+1)^s,\\
&+x^{3s}+x^{3s}(x+1)^s(x^2+x+1)^s-2x^{3s}(x+1)^s)\Big)\\
\frac{B_4^r(x)}{x-1} =&(x-1)^{2s}\Big(\frac{1}{4}(x-1)^{2s}-\frac{1}{4}(x+1)^{2s}+(x^2-1)^s x^s(1-(x+1)^s),\\
&+\frac{1}{2}(x+1)^{2s}x^{2s}(1-(x^2+1)^s)+\frac{1}{2}(x-1)^{2s}x^{2s}(1-(x+1)^s)^2\\
&-(x-1)^{2s}x^{3s}(-(x+1)^s(x^2+x+1)^s+2(x+1)^s-1)\\
&-(x-1)^{2s}x^{6s}(-(x+1)^s(x^2+x+1)^s(x^3+x^2+x+1)^s\\
&+2(x+1)^s(x^2+x+1)^s+(x+1)^{2s}-3(x+1)^s+1)\Big).
\end{aligned}$$

Solution of conjecture for Langlands dual groups

Again $\Gamma = F_r$ the free group of rank r,

Curious equality [Lawton-Muñoz '16]: For all $r, n = 2, 3$

$$E(\mathscr{X}_{F_r} SL_n\mathbb{C}) = E(\mathscr{X}_{F_r} PGL_n\mathbb{C}).$$

They suspected the equality for higher n.

Theorem (Florentino–N.–Zamora (2021))

There are isomorphisms of mixed Hodge structures $H^*(\mathscr{X}_r SL_n) \cong H^*(\mathscr{X}_r PGL_n)$ *and* $H_c^*(\mathscr{X}_r^{irr} SL_n) \cong H_c^*(\mathscr{X}_r^{irr} PGL_n)$. *In particular, their E-polynomial coincide. Moreover for all partition* $[k] \in \mathscr{P}_n$, *we have*

$$E(\mathscr{X}_\Gamma^{[k]} SL_n) = E(\mathscr{X}_\Gamma^{[k]} PGL_n) = (uv-1)^{-r} E(\mathscr{X}_\Gamma^{[k]} GL_n).$$

Idea of proof

- The action $\mathscr{R}_r\mathbb{C}^* \times \mathscr{X}_r GL_n \to \mathscr{X}_r GL_n$ preserve the stratification of GL_n, so we define:

$$\mathscr{X}_r^{[k]} PGL_n := \mathscr{X}_r^{[k]} GL_n / \mathscr{R}_r\mathbb{C}^* = \mathscr{X}_r^{[k]} GL_n / (\mathbb{C}^*)^r \ ,$$

- we define the $[k]$-stratum of $\mathscr{X}_r SL_n$ by restriction of the corresponding one for GL_n:

$$\mathscr{X}_r^{[k]} SL_n := \{\rho \in \mathscr{X}_r^{[k]} GL_n | \det \rho = 1\} \ .$$

- The character varieties $\mathscr{X}_r SL_n$ and $\mathscr{X}_r PGL_n$ can be written as a disjoint unions of of locally closed quasi-projective varieties, labelled by partitions $[k] \in \mathscr{P}_n$

$$\mathscr{X}_r SL_n = \bigsqcup_{[k]\in\mathscr{P}_n} \mathscr{X}_r^{[k]} SL_n, \qquad \mathscr{X}_r PGL_n = \bigsqcup_{[k]\in\mathscr{P}_n} \mathscr{X}_r^{[k]} PGL_n$$

Proposition (Florentino-N. -Zamora (2021))

The fibration

$$\mathscr{R}_r\mathbb{C}^* \to \mathscr{X}_r^{[k]}GL_n \to \mathscr{X}_r^{[k]}PGL_n$$

is special, therefore

$$E(\mathscr{X}_r^{[k]}GL_n) = (uv-1)^r E(\mathscr{X}_r^{[k]}PGL_n)\,,$$

$$\text{and} \quad E(\mathscr{X}_r GL_n) = (uv-1)^r E(\mathscr{X}_r PGL_n)\,.$$

It is hard to prove $E(\mathscr{X}_r GL_n) = (uv-1)^r E(\mathscr{X}_r SL_n)$: we prove the equality by distinguishing between partitions with at least 2 parts and the irreducible case ([k] = [n] of length = 1).

Proposition

If length $[k] \in \mathscr{P}_n$ *is* >1, *then* $E(\mathscr{X}_r^{[k]}GL_n) = (x-1)^r E(\mathscr{X}_r^{[k]}SL_n)$. *Therefore* $E(\mathscr{X}_r^{[k]}SL_n) = E(\mathscr{X}_r^{[k]}PGL_n)$.

Theorem (Florentino-N.-Zamora (2021))

The central action of $\mathbb{Z}_n^r$ on $\mathscr{X}_r^{irr} SL_n$ giving the quotient map

$$\mathscr{X}_r^{irr} SL_n \to \mathscr{X}_r^{irr} PGL_n$$

induces an isomorphism of mixed Hodge structures $H^*(\mathscr{X}_r^{irr} SL_n) \cong H^*(\mathscr{X}_r^{irr} PGL_n)$.

We use differential geometric techniques, taking advantage of the fact that $\mathscr{X}_r^{irr} SL_n$ is a smooth variety and $\mathscr{X}_r^{irr} PGL_n$ is an orbifold (Florentino-Lawton 2014).

- Define $U^*_{r,n} = \mathrm{Hom}^{irr}(F_r, U(n)) \subset U_{r,n} = \mathrm{Hom}(F_r, U(n))$ and similarly $SU^*_{r,n}$, $SU_{r,n}$, $PU^*_{r,n}$, $PU_{r,n}$: we can also get $PU_{r,n}$ and $PU^*_{r,n}$ as finite quotients of $SU_{r,n}$ and $SU^*_{r,n}$:

$$PU_{r,n} = SU_{r,n}/C_{r,n}, \qquad PU^*_{r,n} = SU^*_{r,n}/C_{r,n}.$$

  where $C_{r,n} = \mathrm{Hom}(F_r, \mathbb{Z}_n)$
- Define a *stratification by polystable type* of $U_{r,n}$, $SU_{r,n}$ and $PU_{r,n}$ in complete analogy with the stratifications for GL_n.

Step 1

There are isomorphisms $H^*(SU^*_{r,n}) \cong H^*(PU^*_{r,n})$, $H^*(\mathscr{X}^*_r SU_n) \cong H^*(\mathscr{X}^*_r PU_n)$ and $H^*(\mathscr{X}_r SU_n) \cong H^*(\mathscr{X}_r PU_n)$.

For the stratification $SU_{r,n} = \bigsqcup_{[k]\in\mathscr{P}_n} SU_r^{[k]}$: If length $[k] > 1$, action $C_{r,n} \curvearrowright H^*(SU^{[k]}_{r,n})$ is trivial (construct homotopies to the identity).
Given that $C_{r,n} \subset SU_{r,n}$ connected, action $C_{r,n} \curvearrowright H^*(SU_{r,n})$ is trivial.
The action $C_{r,n} \curvearrowright H^*(SU^*_{r,n})$ is trivial
The quotient map $\pi : SU^*_{r,n} \to PU^*_{r,n} = SU^*_{r,n}/C_{r,n}$ is $PU(n)$-equivariant and the $PU(n)$ action is free on the irreducible representation spaces, then

$$H^*(\mathscr{X}^*_r SU_n) \cong H^*_{PU(n)}(SU^*_{r,n}) \cong H^*_{PU(n)}(PU^*_{r,n}) \cong H^*(\mathscr{X}^*_r PU_n),$$

Step 2

- Use the strong deformation retraction:

$$H^*(\mathscr{X}_r SL_n) \overset{SDR}{\cong} H^*(\mathscr{X}_r SU_n) \overset{Step1}{\cong} H^*(\mathscr{X}_r PU_n) \overset{SDR}{\cong} H^*(\mathscr{X}_r PGL_n).$$

Since the quotient $\mathscr{X}_r SL_n \to \mathscr{X}_r PGL_n$ is algebraic, the above isomorphism H is an isomorphism of mixed Hodge structures.

Step 3

- In the same fashion $H^*(\bigsqcup_{|[k]|\geq 2} \mathscr{X}_r^{[k]} SL_n) \cong H^*(\bigsqcup_{|[k]|\geq 2} \mathscr{X}_r^{[k]} PGL_n)$, as MHS.
- Finally, using the 5-lemma, $H_c^k(\mathscr{X}_r^{irr} SL_n) \cong H_c^k(\mathscr{X}_r^{irr} PGL_n)$.

Again, since the finite quotient $\mathscr{X}_r^{irr} SL_n \to \mathscr{X}_r^{irr} PGL_n$ is algebraic, this implies the isomorphism of mixed Hodge structures on the corresponding compactly supported cohomology groups, and the equality of E-polynomials.

E-polynomial of the SL_4 and PGL_4-character varieties of the free group

Theorem (Florentino-N.-Zamora (2021))

The E-polynomials of the SL_4 (and PGL_4)-character varieties of the free group $\Gamma = F_{s+1}$ are equal to

$$
\begin{aligned}
E(\mathscr{X}_{s+1}SL_4;x) = {} & \frac{1}{24}(x-1)^{3s+3} \\
& +(x-1)^{3s+1}\big[(x+1)^{2s}\frac{x^{2s}}{2}+(x+1)^s(x^{3s}(x^2+x+1)^s-2x^{3s}-x^{2s}+\frac{3x^s}{2})\big] \\
& +(x-1)^{3s+1}\big[x^{3s}+\frac{x^{2s}}{2}-\frac{3x^s}{2}+\frac{11}{24}\big] \\
& +(x-1)^{3s}(x+1)^{2s}(-x^{6s}+\frac{x^{2s}}{2}) \\
& +(x-1)^{3s}(x+1)^s x^{6s}\big[(x^2+x+1)^s(x^3+x^2+x+1)^s-2(x^2+x+1)^s+3\big] \\
& +(x-1)^{3s}(x+1)^s\big[x^{3s}((x^2+x+1)^s-2)-x^{2s}+\frac{x^s}{2}\big] \\
& +(x-1)^{3s}(-x^{6s}+x^{3s}+\frac{x^{2s}}{2}-\frac{x^s}{2}+\frac{1}{2}) \\
& +(x-1)^{2s+2}\frac{(x-1)^{s+1}}{4} \\
& +(x-1)^{2s+1}\frac{(x+1)^s}{2}(-(x+1)^s x^s+x^s-\frac{1}{2}) \\
& +(x-1)^{2s}(x+1)^s\frac{x^s}{2}(1-(x+1)^s) \\
& +(x-1)^{s+1}\big[(x+1)\frac{x}{3}(x^2+x+1)^s+\frac{(x+1)^{2s}}{8}(x^2+2x+2)\big] \\
& +(x-1)^s(x+1)^{2s}\big[\frac{x^{2s+1}}{2}((x^2+1)^s-1)+\frac{x-1}{4}\big] \\
& -\frac{1}{4}(x+1)^{s+1}(x^2+1)^s+\frac{1}{4}(x^3+x^2+x+1)^{s+1}.
\end{aligned}
$$

Corollary

The Euler characteristics of the SL_n and PGL_n character varieties of the free group are

$$\chi(\mathscr{X}_r SL_n) = \chi(\mathscr{X}_r PGL_n) = \varphi(n) n^{r-2}$$

where $\varphi(n)$ is the arithmetic Euler function. The Euler characteristics for the strata of the form $[d^{n/d}] \in \mathscr{P}_n$ are

$$\chi(\mathscr{X}_r^{[d^{n/d}]} SL_n) = \chi(\mathscr{X}_r^{[d^{n/d}]} PGL_n) = \frac{\mu(d)}{d} n^{r-1} \, ,$$

otherwise $\chi(\mathscr{X}_r^{[k]} SL_n) = 0$, where $\mu(n)$ is the arithmetic Möebius function.

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Thank you!