

Cohomological Hall algebras and nonabelian Hodge
isomorphism for stacks
joint work with Ben Davison and
Sebastian Schlegel Mejia.

C smooth projective curve / \mathbb{C} genus g

$$\mathcal{M}_{g,r,d}^B \hookrightarrow \mathcal{M}_{r,d}^{\text{PGL}}(C) \hookrightarrow \mathcal{M}_{r,d}^{\text{dR}}(C)$$

Betti

$$(r, d) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}.$$

r rk d degree

NAHT All 3 spaces are homeomorphic -
in particular, they have isomorphic cohomology / BT homology

better behaved
for singular or
non compact spaces

Questions: ① Can we compare the moduli stacks? non compact spaces -
② Can we at least compare their BT homology?

Today: Yes for ② -

① Betti: $C \ni c$

$$\pi_1(C, c) = \langle x_i y_i, 1 \leq i \leq g \mid \prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1} = 1 \rangle.$$

character variety:

$$\mathcal{M}_{g,r,0}^B = \left\{ M_i, N_i \in \text{GL}_r^{2g} \mid \prod M_i N_i T_i^{-1} N_i^{-1} = I \right\}$$

algebraic variety.

GL_r

character stack $\mathcal{M}_{g,r,0}^B = [R_{g,r,0}^B / \text{GL}_r]$.

\ studying the GL_n -equivariant
geometry of $R_{g_{m,0}}^B$.

(2) Dolbeault c.

(\mathcal{F}, θ)
 $\uparrow v.b.$ Higgs field $\theta: \mathcal{F} \rightarrow \mathcal{F} \otimes K_C$ can. bundle
 r, d G_C - linear.

stability : $g \subset \mathcal{F}$, inequality.

(g72) $M_{r,d}^{\text{Dol}}(C)$ alg. variety parametrising
polystable Higgs bundles r, d ,

- * irreducible
- * smooth when (r, d) coprime
- * $2(g-1)r^2 + 2$
- * $M_{r,d}^{\text{Dol}}(C) = \underbrace{R_{r,d}^{\text{Dol}}(C)}_{\text{parametrises framed Higgs bundles.}} // GL_r$
- * $M_{r,d}^{\text{Dol}}(C) = \left[\underbrace{R_{r,d}^{\text{Dol}}(C)}_{\text{parametrises framed Higgs bundles.}} / GL_r \right]$.

(3) de Rham. $d = 0$

(\mathcal{F}, ∇)
 $\uparrow v.b.$ connection $\nabla: \mathcal{F} \rightarrow \mathcal{F} \otimes K_C$
 Leibniz rule.

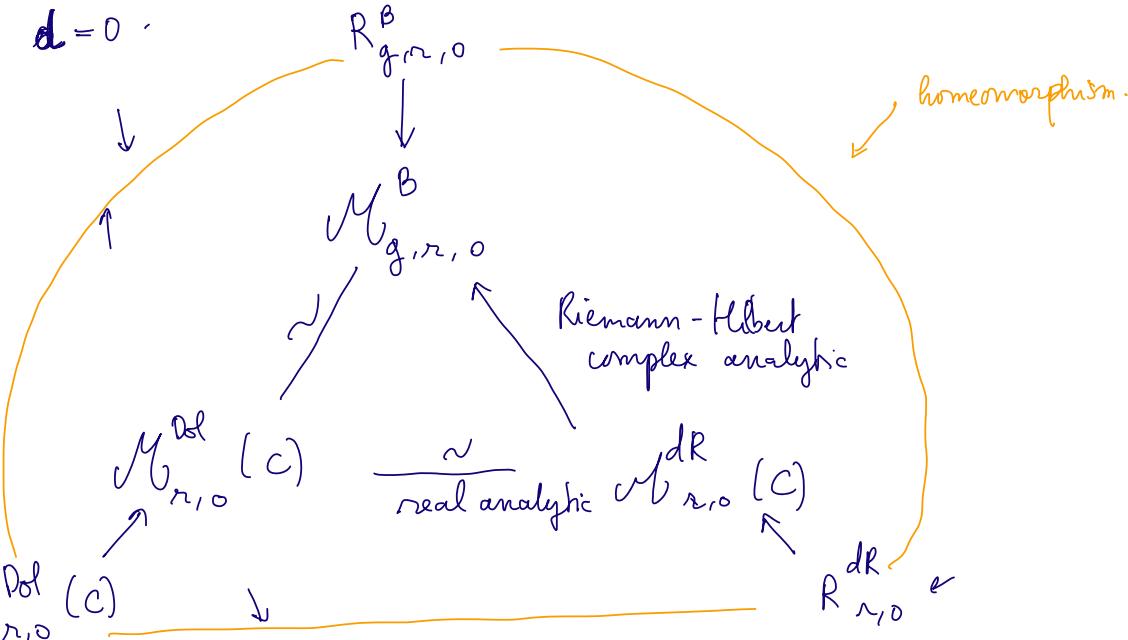
$$M_{r,0}^{dR}(C) = R_{r,0}^{dR}(C) // GL_r$$

$$M_{r,0}^{dR}(C) = \left[R_{r,0}^{dR}(C) / GL_r \right].$$

N A H T

Hitchin
Corlette
Donaldson
Simpson

$d = 0$



Riemann-Hilbert
complex analytic

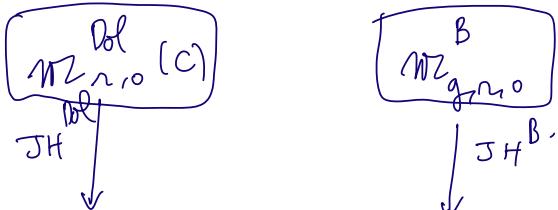
\sim
real analytic

/ are abelian - theoretic GL_n -equivariant bijections (Simpson).

not homeo. (counterexample by Simpson).

$$H_*^{B\cap}(\mathcal{M}_{g,n,0}^B) \cong H_*^{B\cap}(\mathcal{M}_{n,0}^{dR}(C)) .$$

Question: How to deal with the remaining arrows?



$$\mathcal{M}_{n,0}^{Dol}(C) \xrightarrow[\psi]{} \mathcal{M}_{g,n,0}^B$$

$$\pi \quad \quad \quad \pi$$

/ vir is some coh. shift.

$$\mathcal{A}_n^{\text{Dol}} := \text{JH}_{\infty}^{\text{Dol}} \mathbb{D} Q_{M_{n,0}^{\text{Dol}}}^{\text{vir}}(c) \in \mathcal{D}_c^+(\mathcal{M}_{n,0}^{\text{Dol}}(c))$$

S1

$$\mathcal{A}_n^B := \text{JH}_{\infty}^B \mathbb{D} Q_{M_{g,n,d}}^{\text{vir}} \in \mathcal{D}_c^+(\mathcal{M}_{g,n,d}^B(c))$$

we want to compare $\mathcal{A}_n^{\text{Dol}}$ and \mathcal{A}_n^B .

$$\mathcal{A}^{\text{Dol}} = \bigoplus_{n \geq 1} \mathcal{A}_n^{\text{Dol}}$$

$$\mathcal{M}^{\text{Dol}}(c) := \bigsqcup_{n \geq 1} \mathcal{M}_{n,0}^{\text{Dol}}(c)$$

$$\mathcal{A}^B = \bigoplus_{n \geq 1} \mathcal{A}_n^B.$$

$$\mathcal{M}_g^B := \bigsqcup_{n \geq 1} \mathcal{M}_{g,n,0}^B(c).$$

other (Davison - H - Schlegel Mejia)

① we have a cohomological Hall algebra structure on \mathcal{A}^Δ

$$\Delta \in \{\text{Dol}, B\}$$

② $B\mathcal{P}\mathcal{Y}_{\text{Alg}}^\Delta := \text{P}H^0(\mathcal{A}^\Delta) \in \text{Perf}(\mathcal{M}^\Delta) \leftarrow \text{MHM.}$
is an algebra object.

③ $B\mathcal{P}\mathcal{Y}_{\text{Alg}}^\Delta = \mathcal{T}(\mathcal{B}\mathcal{P}\mathcal{Y}_{\text{Lie}}^\Delta)$

\uparrow
enveloping alg

NAHT
DE complexes are topological invariants.

$$\mathcal{B}\mathcal{P}\mathcal{Y}_{\text{Lie}}^\Delta := \text{Free}_{\text{Lie}} \left(\bigoplus_n \mathcal{D}\mathcal{E}(V_{n,0}^\Delta) \right).$$

④ PBW-iso:

$$\text{Sym} \left(\mathcal{B}\mathcal{P}\mathcal{Y}_{\text{Lie}}^\Delta \otimes \text{H}^*(BC^*) \right) \xrightarrow{\sim} (\mathcal{A}^\Delta)$$

$\mathbb{C}[u]$

monoidal structure $\mathcal{F}, \mathcal{G} \in \mathcal{D}_c^+(\mathcal{M}^\Delta)$

$$\mathcal{F} \boxtimes \mathcal{G} := \bigoplus_x (\mathcal{F} \otimes \mathcal{G})$$

$$\oplus: \mathcal{M}^\Delta \times \mathcal{M}^\Delta \rightarrow \mathcal{M}^\Delta$$

Corollary: $\boxed{A^B \cong A^{\text{Dol}}} \in \mathcal{D}_c^+(\mathcal{M}^\Delta)$ $\Delta \in \{B, \text{Dol}\}$.

$$\pi_{*} A^B \cong \pi_{*} A^{\text{Dol}}$$

$$\boxed{H_*^{\text{BN}}(\mathcal{M}_{\mathbb{G}}^B) \cong H_*^{\text{BN}}(\mathcal{M}_{\mathbb{G}}^{\text{Dol}}(C))}.$$

Essential ingredient: • local description of the maps $JH^\Delta: M^\Delta \rightarrow \mathcal{M}^\Delta$.

- comes from the fact that, in a precise sense,
the categories $\text{Higgs}^{M-\text{sst}}(C)$ are 2CY Abelian categories.
- $\text{Rep } \pi_1(C, c)$

$$x \in \mathcal{M}^\Delta \quad \mathcal{F} = \bigoplus_{i=1}^s \mathcal{F}_i^{m_i} \uparrow \quad \mathcal{F}_i \text{ are pairwise nonisomorphic, } m_i > 0. \quad \underline{\text{simple.}}$$

$$\underline{\mathcal{E}} = \{ F_1, \dots, F_s \}$$

$\overline{\mathcal{Q}}_{\underline{\mathcal{E}}} = \text{Ext - quiver of } \underline{\mathcal{E}}$.

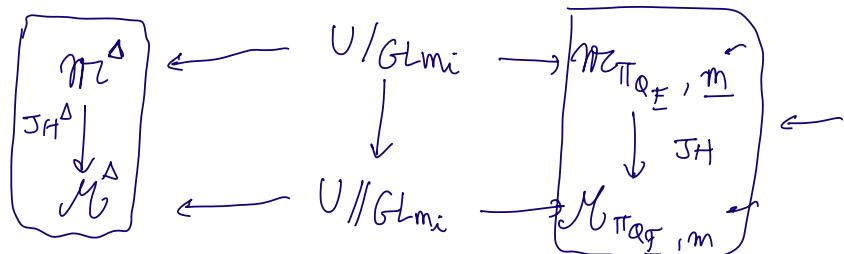
$$\begin{matrix} \text{vertices } \underline{\mathcal{E}}; \\ \text{edges } \underline{\mathcal{E}}; \end{matrix} \# \{ F_i \rightarrow F_j \} = \dim \text{Ext}^1(F_i, F_j).$$

$\overline{\mathcal{Q}}_{\underline{\mathcal{E}}}$ is the double of some quiver $\mathcal{Q}_{\underline{\mathcal{E}}}$

$$\mathcal{Q} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad \rightsquigarrow \quad \overline{\mathcal{Q}} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

$\mathbb{T} \mathcal{O}_{\underline{\mathcal{E}}} = \text{preprojective algebra of } \mathcal{Q}_{\underline{\mathcal{E}}}$.

2CY categories
Ben Davison.



s.t. horizontal maps are étale.

$$\overline{Q} \text{ mod } \pi \overline{\alpha} = \begin{matrix} \infty \\ \infty \end{matrix}$$

↑
oriented
graph

$$\Pi_Q = \frac{\infty}{\infty}$$

$$f = [a, a^*] + [b, b^*] + [c, c^*]$$
$$\in \infty$$

$$\overline{\alpha} = \begin{matrix} a^* \\ b \\ c \\ b^* \\ c^* \end{matrix}$$