An algebraic construction of K-moduli space

Chenyang Xu

Princeton University

VBAC Webinar Series, 2020 Nov. 23

- Let X be a Fano manifold. Tian first defined, and later Donaldson put in algebraic terms, the notions of K-stability for X as the positivity (or non-negativity) of $\mathrm{Fut}(\mathcal{X}, \mathcal{L})$ for all test configurations (TC) $(\mathcal{X}, \mathcal{L})$.
- Here a TC $(\mathcal{X}, \mathcal{L})/\mathbb{A}^1$ is a flat $\mathbb{G}_m$-equivariant degeneration of $(X, -K_X)$, with $\mathcal{L}$ an ample $\mathbb{Q}$-line bundle.

- E is on a birational model $\mu\colon Y \to X$. Let $A_X(E)$ be the log discrepancy of E, i.e.

$$\mu^* K_X + \sum_i A_X(E_i) \cdot E_i = K_Y + \sum_i E_i.$$

- X is klt if $A_X(E) > 0$.
- $S_X(E)$ is the expected vanishing order, i.e.,

$$S_X(E) = \frac{1}{(-K_X)^n} \int_0^\infty \mathrm{vol}(\mu^*(-K_X) - tE)dt.$$

- $\beta_X(E) := A_X(E) - S_X(E)$ and $\delta(X) := \inf_E \frac{A_X(E)}{S_X(E)}$

Theorem (Fujita-Li Valuative Criterion, 2016)

For a Fano variety X,

$$\mathrm{Fut}(\mathcal{X}, \mathcal{L}) \geq 0 \ \forall \mathcal{X} \iff \beta_X(E) \geq 0 \ (\forall E/X) \Big(\iff \delta(X) \geq 1 \Big).$$

- Let $R=\bigoplus_{m\in\mathbb{N}} R_m=\bigoplus_{m\in\mathbb{N}} H^0(-mK_X)$.
- A (linearly bounded multiplicative) filtration $\mathcal{F}^\lambda$ ($\lambda\in\mathbb{R}$) on R, compatible with the grading, is assumed to satisfy
  1. $\mathcal{F}^\lambda R_m=\bigcap_{\lambda'<\lambda}\mathcal{F}^{\lambda'}R_m$.
  2. $\mathcal{F}^{\lambda_0}R_{m_0}\cdot\mathcal{F}^{\lambda}R_m\subset\mathcal{F}^{\lambda_0+\lambda}R_{m_0+m}$.
  3. there exist $e_-,e_+\in\mathbb{R}$, $\mathcal{F}R_m^{e_-m}=R_m$ and $\mathcal{F}R_m^{e_+m}=0$.
- Example: a divisorial valuation v, gives a filtration $\mathcal{F}_v^\lambda R_m=\{f\in R_m\mid v(f)\ge\lambda\}$. The graded ring is integral.
- Example: Let $(\mathcal{X},\mathcal{L})$ be a TC and m sufficiently divisible. Then we can define a filtration $\mathcal{F}^\lambda R_m=\{s\in R_m\mid \text{the rational section } t^{-\lambda}\bar{s}\in H^0(\mathcal{X},m\mathcal{L})\}$, where $\bar{s}$ is the section $s\times\mathbb{A}^1$. This filtration is finitely generated.

  Such filtrations are $\mathbb{Z}$-valued and finitely generated.

- For $S_m(\mathcal{F}) := \frac{1}{m \dim R_m} \sum_{\lambda\in\mathbb{R}} \lambda \dim \mathrm{Gr}^{\lambda}_{\mathcal{F}} R_m$.
- Define $S(\mathcal{F}) := \lim_{m\to\infty} S_m(\mathcal{F})$.
- The base ideals $\mathfrak{b}_{m,n} : \mathfrak{b}\big(\mathcal{F}^n H^0(-mK_X) \to H^0(-mK_X)\big)$. Let $\mathfrak{b}^t_\bullet = \{\mathfrak{b}_{m,mt}\}_m$.

Theorem (Zhuang-X.)

Let the log canonical slope $\mu(\mathcal{F}) := \sup\{t \mid \mathrm{lct}(X, \mathfrak{b}^t_\bullet) \geq 1\}$, and

$$\beta(\mathcal{F}) := \mu(\mathcal{F}) - S(\mathcal{F}).$$

X is K-semistable if and only if $\beta(\mathcal{F}) \geq 0$ for any $\mathcal{F}$.

The main research topics in K-stability theory:

1. understanding notions of K-stability;
2. K-moduli;
3. verify a given Fano is K-stable.

Definition

Fix $n = \dim X$, $V = (-K_X)^n$

1. (K-moduli stack) A finite type stack $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ parametrizes families of such K-semistable Fano varieties;
2. (K-moduli space) A good moduli space $\mathfrak{X}^{\mathrm{Kss}}_{n,V} \to X^{\mathrm{Kps}}_{n,V}$ pointwisely parametrizes K-polystable Fano varieties.

Étale locally around any point on a good moduli space, the morphism has the form $[\mathrm{Spec}(A)/G] \to \mathrm{Spec} A^G$ for a reductive G.

Theorem

The K-moduli stack $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ exists as a global quotient stack, and it admits a good moduli space $X^{\mathrm{Kps}}_{n,V}$ which is separated.

Theorem

The K-moduli stack $\mathfrak{X}^{\mathrm{Kss}}_{n,V}$ exists as a global quotient stack of finite type.

Theorem (Boundedness, Jiang, X.-Zhuang)

Fix $v, \delta > 0$. All n-dimensional Fano varieties X with $(-K_X)^n \geq v$ and $\delta(X) \geq \delta$ form a bounded set.

Theorem (Openness, X., Blum-Liu-X.)

Let $X \to S$ be a family of Fano varieties. The locus where the fiber X_s is K-semistable is open.

In the proofs of the theorems above, it requires some deep boundedness type results in birational geometry.

- For any K-semistable Fano varieties with fixed dimension n and volume V can be embedded in $\mathbb{P}^N$ where $N = N(n, V)$ by boundedness.
- Then by the local KSB theory, there exists a locally closed subscheme $Z \subset \mathrm{Hilb}(\mathbb{P}^N)$ of a finite type Hilbert scheme which parametrizes klt Fano varieties with a fixed dimension n and volume V, containing all K-semistable ones.
- K-semistability is an open condition. So there exists an open subscheme $Z^\circ \subset Z$ parametrizing families of K-semistable Fano varieties.
- Then $\mathfrak{X}^{\mathrm{Kss}}_{n,V} = [Z^\circ/\mathrm{PGL}(N+1)]$.

Theorem (Li-Wang-X., Blum-X., Alper-B.-Halpern-Leistner-X.)

The K-moduli space $\mathfrak{X}^{\mathrm{Kss}}_{n,V} \to X^{\mathrm{Kps}}_{n,V}$ *exists as a separated algebraic space.*

Theorem (Blum-X.)

Let X/C and X'/C be two family of Fano varieties, and $X \times_C C^\circ \cong X' \times_C C^\circ$ where $C^\circ = C \setminus \{0\}$. If X_0 and X'_0 are K-semistable, then they degenerate to the same K-semistable Fano variety Y.

- Let $\mathcal{R} := \bigoplus_m H^0(X, -mK_X)$. There is a fltration $\mathcal{F}$ on $\mathcal{R}$ given by the vanishing order of X_0'.
- The image $\mathcal{F}^\bullet\mathcal{R} \to \mathcal{R} \to R := \bigoplus_m H^0(X_0, -mK_{X_0})$ gives a filtration on R. We do the same for $\mathcal{R}' = \bigoplus_m H^0(X', -mK_{X'})$.
- Let $a = A_{X,X_0}(X_0')$ and $a' = A_{X',X_0'}(X_0)$.
- (Key observation) There is an isomorphism

$$R_0 := \mathrm{Gr}_{\mathcal{F}} R \cong \mathrm{Gr}_{\mathcal{F}'} R',$$

  and by tracking the degree for this morphism we have $(a + a') - (S_{X_0}(\mathcal{F}) + S_{X_0'}(\mathcal{F}')) = 0$.
- Moreover, $\mu_{X_0}(\mathcal{F}) \geq a$ and $\mu_{X_0'}(\mathcal{F}') \geq a'$.
- Thus $\mu(\mathcal{F}) = a$, $\mu(\mathcal{F}') = a'$ and $\beta(\mathcal{F}) = \beta(\mathcal{F}') = 0$, since X_0 and X_0' are K-semistable.
- Combining with MMP constructions shows that $\mathrm{Gr}_{\mathcal{F}} R$ is finitely generated. It yields degenerations of $X_0, X_0' \rightsquigarrow Y := \mathrm{Proj}(\mathrm{Gr}_{\mathcal{F}} R)$.
- One can show the total family is normal, and its Futaki invariant vanishes, which implies Y is a K-semistable Fano variety.

(Alper-Blum-Halpern-Leistner-X.) Reinterpretation using A-HL-Heinloth's general valuative criterion: S-completeness.

- Let A be a DVR, $\mathrm{ST}_A = \left[\mathrm{Spec}\left(A[s,t]/\pi - st\right)/\mathbb{G}_m\right]$ with $\mathbb{G}_m$-action $s \to \mu s$ and $t \to \mu^{-1}t$. Let $0 = \left[(s=t=0)/\mathbb{G}_m\right]$. Then $\mathrm{ST}_A^\circ := \mathrm{ST}_A \setminus 0 = \mathrm{Spec} R \bigcup_{\mathrm{Spec} K} \mathrm{Spec} R$ is a double pointed line.
- We obtain a family $\mu^\circ \colon \mathfrak{X}^\circ := X \bigcup_{X^\circ \cong X'^\circ} X' \to \mathrm{ST}_A^\circ$. The pushforward $\mathfrak{R}$ of $\mathfrak{R}^\circ := \bigoplus_m \mu_*^\circ(-mK_{\mathfrak{X}^\circ})$ is a flat $\mathcal{O}_{\mathrm{ST}_A}$-algebra.
- Its central fiber is isomorphic to R_0. Thus $\mathfrak{R}$ is a finitely generated $\mathcal{O}_{\mathrm{ST}_A}$-algebra.
- Taking Proj, we extend the family $\mathfrak{X} \to \mathrm{ST}_A$.
- Summary: Any $\mathrm{ST}_A^\circ \to \mathfrak{X}_{n,V}^{\mathrm{Kss}}$ can be uniquely extended to a morphism $\mathrm{ST}_A \to \mathfrak{X}_{n,V}^{\mathrm{Kss}}$, i.e. $\mathfrak{X}_{n,V}^{\mathrm{Kss}}$ satisfies S-completeness.

- If we glue two trivial families $X \times \mathrm{Spec} A$ over a complete DVR A, by an element $g \in \mathrm{Aut}(X)(K)$ where X is K-polystable.
- The above argument shows we get a torsor of $\mathfrak{X}$ over ST_A.
- Another torsor is given by $\left[X \times \mathrm{Spec}\left(A[s,t]/\pi - st\right)\right]/\mathbb{G}_m$, where $\lambda\colon \mathbb{G}_m \to \mathrm{Aut}(X)$ is induced by the fiber over 0.
- $\mathrm{Isom}(\mathfrak{X}, X_{\mathrm{ST}_A}) \to \mathrm{ST}_A$ has a section of 0, and it is formal smooth, so it has a section of ST_A.
- That implies $g = a \cdot \lambda \cdot b$, where $a \in \mathrm{Spec}(R)$, i.e. the Iwahori decomposition holds for $\mathrm{Aut}(X)$.

Theorem (Alper-Blum-Halpern-Leistner-X.)

For K-polystable Fano variety X, $\mathrm{Aut}(X)$ is reductive.

A-HL-Heinloth's general valuative criterion:

S-completeness + Θ-reductivity

for the existence of a separated good moduli space.

Conjecture

The moduli space $X_{n,V}^{\mathrm{Kps}}$ is proper and projective.

- In the smoothable case (where the construction was established by Li-Wang-X. using analytic results), the properness is known (by a result of Donaldson-Sun);
- (X.-Zhuang) The CM line bundle is ample on this proper component.

Conjecture (Optimal Degeneration Conjecture)

If $\delta(X) \leq 1$, then there exists a divisor E, such that $\delta(X) = \frac{A_X(E)}{S_X(E)}$.

- (Blum-Halpher-Leistner-Liu-X.) Assuming the optimal degeneration conjecture, then the abstract Langton argument can be used to prove $X^{\mathrm{Kps}}_{n,V}$ is proper.
- The optimal degeneration conjecture implies

$$\text{(K-stability)=(uniform K-stability)},$$

  and completes the solution of Yau-Tian-Donaldson Conjecture for all (possibly singular) Fano varieties.
- (X.-Zhuang) (A stronger version of) the optimal degeneration conjecture also implies that $X^{\mathrm{Kps}}_{n,V}$ is projective, or more precisely, the CM line bundle is ample on $X^{\mathrm{Kps}}_{n,V}$.

Thank you very much!