K-stability and scalar curvature

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We are interested in the question: given a compact Kähler manifold (X, ω_0) is there an “optimal” metric in the Kähler class $[\omega_0] \in H^2(X)$?

The question has intrinsic differential geometric interest and is also fundamentally related to **moduli problems**, since we have a moduli space of all Riemannian metrics on X modulo diffeomorphism.

The metrics in the fixed Kähler class are given by Kähler potentials

$$\omega_\phi = \omega_0 + i\overline{\partial}\partial\phi,$$

so one expects that the right general condition should involve one scalar equation.
The obvious choice is constant scalar curvature (CSC):

$$S(\omega_\phi) = \text{constant}. \quad (1)$$

The constant is determined by $c_1(X), [\omega_0]$.
This equation (1) is a fourth order PDE for the potential ϕ.

When $\dim_{\mathbf{C}} X = 1$ it is well-known that there is a unique solution: a metric of constant Gauss curvature.

- If X has a non-zero holomorphic vector fields "extremal metrics" can be the appropriate thing to consider, but we will not discuss these today.
- The Scalar curvature S is the contraction of the Ricci $(1,1)$ form ρ: i.e. $S = \Lambda\rho$ where $\Lambda : \Omega^{1,1} \to \Omega^0$. Then $\overline{\partial}^* \rho = i\overline{\partial} S$ so the CSC condition is equivalent to ρ being harmonic. If $c_1(X) = \lambda[\omega_0]$ it follows that $\rho = \lambda\omega_0$ i.e. the metric is *Kähler-Einstein*. When $\lambda \leq 0$ existence was proved by Yau and Aubin-Yau in the 1970's.

THE MAIN MESSAGE:

There is a “Kobayashi-Hitchin” setting for the CSC condition, partly conjectural (the “YTD conjecture”), but established in some important cases.

The overall structure is closely analogous to that for stable vector bundles and Hermitian Yang-Mills connections etc., going back to Narasimhan and Seshadri’s result for bundles over algebraic curves.

The unifying concept is that of a “moment map”.

Suppose that a Lie group G acts on a symplectic manifold (Z, Ω). A moment map for the action is a map $\mu : Z \to \mathrm{Lie}(G)^*$ such that at each point $z \in Z$ the derivative

$$d\mu_z : TZ_z \to \mathrm{Lie}(G)^*$$

is equal to the transpose of the infinitesimal action $\mathrm{Lie}G \to TZ_z$ under the isomorphism $TZ = T^*Z$ defined by Ω.
We also require that μ is G-equivariant.

Now suppose that Z is Kähler and G has a complexification G^c which acts holomorphically on Z.
The principle of “equality of symplectic and complex quotients” is that

$$\mu^{-1}(0)/G = Z^s/G^c, \tag{2}$$

where $Z^s \subset Z$ is a suitable open set of stable points.

In other words, we can solve the equation $\mu(g(z_0)) = 0$ for $g \in G^c$ if and only if z_0 is stable, and the solution is unique up to the action of G.

The prototype situation is when $Z = \mathbf{CP}^n$ and G act via a unitary representation on $\mathbf{C}^{n+1}$. A point $z_0 \in \mathbf{CP}^n$ is stable if the G^c orbit of a lift $\tilde{z}_0 \in \mathbf{C}^{n+1}$ is closed.

The existence of a solution to the equation $\mu(g(z_0)) = 0$ follows from minimising the Kempf-Ness function:

$$\mathcal{F}(g) = \log |g(\tilde{z}_0)|^2.$$

If $u \in G$ then $\mathcal{F}(ug) = \mathcal{F}(g)$ so we can regard $\mathcal{F}$ as a function on G^c/G, which is a symmetric space of nonpositive type.

Example For $G = SU(2)$, the symmetric space G^c/G is 3-dimensional hyperbolic space.

The uniqueness of the solution follows from the fact that $\mathcal{F}$ is *convex* along geodesics in G^c/G.

A geodesic ray $\gamma : [0, \infty) \to G^c/G$ corresponds to an analytic 1-parameter subgroup in G^c. Define

$$w(\gamma) = \lim_{t\to\infty} \frac{d}{dt} \left(\mathcal{F}(\gamma(t))\right).$$

The **Hilbert-Mumford criterion** states that the orbit is stable if $w(\gamma) > 0$ for all rational geodesic rays (corresponding to algebraic 1-parameter subgroups $\mathbf{C}^* \to G^c$).

Going back to general setting of G, G^c, Z, Ω; for each $z_0 \in Z$ we can define a function $\mathcal{F} : G^c/G \to \mathbf{R}$ (up to a arbitrary constant) by the formula

$$\delta\mathcal{F} = \langle \mu(g(z_0)), i\delta g g^{-1} \rangle.$$

The principle (2) is the statement that $\mathcal{F}$ has a minimum if and only if z_0 is stable.

For finite dimensional manifolds the principle (2) can be made into a precise general theorem.

We are interested in infinite dimensional examples where it serves as a guide to what might be proved.

Recall the Atiyah-Bott interpretation of the Narasimhan-Seshadri theorem.
Σ is a compact oriented surface and $E \to \Sigma$ is a fixed Hermitian vector bundle. For simplicity suppose $c_1(E) = 0$.
$\mathcal{A}$ is the space of unitary connections on E and $\mathcal{G}$ is the group of unitary automorphisms.
A tangent vector to $\mathcal{A}$ is a bundle-valued 1-form a. In terms of covariant derivatives

$$d_{A+a} = d_A + a.$$

There is a symplectic form

$$\Omega(a,b) = \int_\Sigma \mathrm{Tr}(a \wedge b),$$

and $A \mapsto F(A)$ is a moment map for the action.
So $\mu^{-1}(0)/\mathcal{G}$ is the moduli space of flat unitary connections.

Now let Σ have a complex structure. We can write

$$d_A = \overline{\partial}_A + \partial_A$$

The operator $\overline{\partial}_A$ defines a holomorphic structure on E.
The group $\mathcal{G}^c$ of general linear automorphisms of E acts on $\mathcal{A}$ via its action on the $\overline{\partial}$-operators

$$d_{g(A)} = g\overline{\partial}_A g^{-1} + (g^*)^{-1}\partial_A g^*.$$

The quotient set $\mathcal{A}/\mathcal{G}^c$ is the set of isomorphism classes of holomorphic structures on E.

So the principle (1) holds true with $\mathcal{A}^s \subset \mathcal{A}$ the set of Mumford stable holomorphic structures.

In this case $\mathcal{G}^c/\mathcal{G}$ is the set of Hermitian metrics on E.
Two points of view

- Fix the Hermitian structure and vary the $\overline{\partial}$-operator by the action of $\mathcal{G}^c$;
- Fix the holomorphic structure and vary the Hermitian metric on E.

Go back to Kähler metrics.
Fix a symplectic manifold (V^{2n}, ω) and let $\mathcal{J}$ be the set of compatible almost complex structures, the sections of a bundle over V with fibre $M = Sp(n, \mathbf{R})/U(n)$. This has an $Sp(n, \mathbf{R})$-invariant Kähler structure I_M, Θ_M.
We get an induced (formal) Kähler structure on $\mathcal{J}$.

$$I(\delta J)(x) = I_M(\delta J(x))$$

$$\Omega(\delta_1 J, \delta_2 J) = \int_V \Theta_M(\delta_1 J, \delta_2 J)\omega^n.$$

The group $\mathcal{G}$ of "exact" symplectic diffeomorphisms of V acts on $\mathcal{J}$.
The Lie algebra of $\mathcal{G}$ is given by the vector fields generated by the Hamiltonian construction so it can be identified with functions of integral 0.
The scalar curvature, modulo constants, is the moment map for the action (Fujiki).
BUT we do not have a complexified group.

In general: suppose that G acts on a complex manifold Z. We say that that a formal complexification of the action is an equivalence relation $\sim$ on Z such that each equivalence class is a submanifold and if $O_z^{\mathbf{C}}$ is the equivalence class of z we have

$$(TO_z^{\mathbf{C}})_z = (TO_z)_z + i(TO_z)_z$$

where O_z is the G-orbit of z.
The principle (2) makes sense for formal complexifications.

Restrict attention to the subset $\mathcal{J}^{\rm int} \subset \mathcal{J}$ of integrable structures. There is an equivalence relation $J_1 \sim J_2$ if and only if $(V, J_1), (V, J_2)$ are equivalent complex manifolds.
This is a formal complexification of the action.
So we get the same picture as before.

If we fix a complex structure J and write $(V, J, [\omega]) = (X, [\omega_0])$ we interpret $\mathcal{G}^g/\mathcal{G}$ as the space $\mathcal{H}$ of Kähler metrics on X in the class $[\omega_0]$.

$\mathcal{H}$ has the structure of an infinite-dimensional Riemannian symmetric space of non-positive curvature (Mabuchi). The metric is

$$\|\delta\phi\|^2 = \int_X \delta\phi^2 \omega_\phi^n.$$

This makes $\mathcal{H}$ a symmetric space in the sense that its curvature is covariant constant. The sectional curvature is given by the usual formula:

$$K(\delta_1\phi, \delta_2\phi) = -\frac{1}{4}|\{\delta_1\phi, \delta_2\phi\}|^2.$$

Geodesics in the infinite dimensional space $\mathcal{H}$ also make sense. Think of a path ϕ_t in $\mathcal{H}$ as a function Φ on $X \times (a,b) \times S^1 \subset X \times \mathbf{C}^*$ then $\Omega_\Phi = \omega_0 + i\overline{\partial}\partial\Phi$ is a $(1,1)$-form on this subset of $X \times \mathbf{C}^*$ and the geodesic equation is

$$\Omega_0^{n+1} = 0.$$

Geometrically, this says that the null spaces of Ω_Φ give a foliation of this subset of $X \times \mathbf{C}^*$ with holomorphic leaves. BUT: to have a good theory of geodesics it is necessary to work with functions which are not smooth and the theory becomes deep and subtle.

The Mabuchi functional $\mathcal{F} : \mathcal{H} \rightarrow \mathbf{R}$ is defined by the same formula as above. It is convex on geodesics in $\mathcal{H}$ and the search for a CSC metric is the problem of finding a critical point (in fact a minimum) of $\mathcal{F}$.

Notions of stability based on numerical criteria.

- Analytic: for all geodesic rays γ in $\mathcal{H}$ the asymptotic derivative $w(\gamma) > 0$.
- K-stability: (for X a complex projective manifold), the Futaki invariant $\mathrm{Fut}(\mathcal{X}) > 0$ for all $\mathbf{C}^*$-equivariant degenerations $\mathcal{X}$ of X.

Futaki invariant and degenerations. Given (X, L) with $c_1(L) = [\omega]$. Consider a family $\pi : \mathcal{X} \to \mathbf{C}$ with a $\mathbf{C}^*$ action on $\mathcal{X}$ covering the standard action on $\mathbf{C}$ and an equivariant $\mathbf{Q}$-line bundle $\mathcal{L} \to \mathcal{X}$, ample on the fibres, such that $(\pi^{-1}(t), \mathcal{L}|_{\pi^{-1}(t)})$ is equivalent to (X, L) for $t \neq 0$.

The Futaki invariant $\mathrm{Fut}(\mathcal{X})$ is defined by the $\mathbf{C}^*$-action on $X_0 = \pi^{-1}(0)$. If X_0 is smooth it can be defined as follows. Choose any S^1 invariant metric ω on X_0 in the given Kähler class. The S^1 action is generated by a Hamiltonian function H and

$$\mathrm{Fut} = \int_{X_0} S(\omega) H.$$

When X_0 is singular the invariant can be defined from the asymptotics of the weights of the action on $H^0(X_0, \mathcal{L}^k)$ as $k \to \infty$ or as the weight of the action on the "CM line" associated to $(X_0, \mathcal{L})$.

Comparison with Mumford stability.

Let E be a rank 2 holomorphic vector bundle over Σ. We can consider $\mathbf{C}^*$ equivariant degenerations $\mathcal{E} \to \Sigma \times \mathbf{C}$. Where the action has determinant 1 on the fibres over $\Sigma \times \{0\}$. These are classified by a line subbundle $L \subset E$ and a weight $\lambda > 0$. On the central fibre the bundle splits as $L \oplus L'$ and the action has weight λ on L'. The analogue of the Futaki invariant is the pairing between the curvature of a connection over the central fibre and the generator of the action. This is

$$\lambda(\mathrm{deg} L' - \mathrm{deg} L) = \lambda(\mathrm{deg} E - 2\mathrm{deg} L).$$

The numerical criterion is the familiar one

$$\deg(L) < \deg(E)/2.$$

There is another side of the story in which one considers metrics induced by projective embeddings $X \to \mathbf{P}(H^0(X, L^k)^*)$ for $k >> 0$. This connects to "asymptotic Chow stability". But we will not go into this today.

Toric manifolds

When X is a toric manifold, with an action of $T^c = (\mathbf{C}^*)^n$ and T-invariant metrics the constructions can be made more explicit.

On the open orbit $T^c \subset X$ take standard co-ordinates $z_i \in \mathbf{C}^*$.

A T-invariant Kähler potential is a *convex* function $\psi(y_1, \dots y_n)$ where $y_i = \log |z_i|$. More invariantly, ψ is a convex function on $\mathrm{Lie}(T)$.

The **Legendre transform** of ψ is a convex function u on the moment polytope $P \subset \mathrm{Lie}(T)^*$ corresponding to X.

u is defined by the condition that

$$u(x) + \psi(\xi) = \langle x, \xi \rangle$$

when $x = d\psi(\xi)$.

The open orbit in X can be identified with $\mathrm{int} P \times T$. The symplectic form is $\omega = \sum dx_i d\theta_i$ and the metric defined by u is

$$\sum u_{ij} dx_i dx_j + u^{ij} d\theta_i d\theta_j$$

where $(u_{ij}) = \nabla^2 u$ and (u^{ij}) is the inverse matrix.

So when the second derivatives of u are large the metric is large in the P direction and small in the T direction.

The integer lattice in $\mathrm{Lie}(T)^*$ fixes a volume form $d\sigma$ on the codimension 1 faces of ∂P.
We define a linear functional L on functions on P by

$$L(f) = \int_{\partial P} f d\sigma - A \int_P f d\mu,$$

where A is chosen so that $L(1) = 0$.

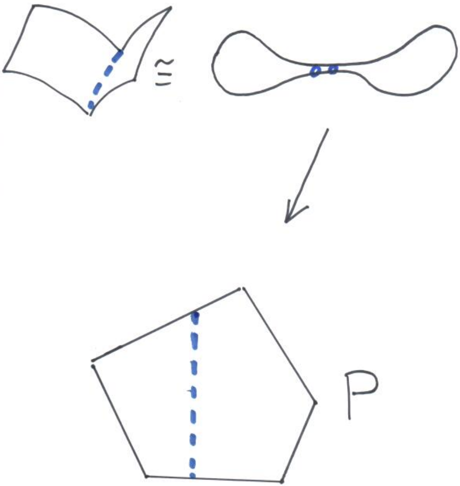
The Mabuchi functional is

$$\mathcal{F}(u) = -\int_P \log\det(\nabla^2 u) + L(u). \tag{3}$$

The CSC equation is

$$\sum_{ij} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = -A.$$

Geodesic rays correspond to lines $\gamma_t = u_0 + tf$ where f is convex.
The asymptotic derivative is $w(\gamma) = L(f)$.
T^c invariant degenerations of X correspond to rational piecewise linear functions f on P and the Futaki invariant is $L(f)$.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Progress towards a “Kobayashi-Hitchin correspondence”

The “easy” direction: *CSC* $\mathrm{metric} \Rightarrow \mathrm{K-stable}$ and uniqueness of CSC metrics are known (Stoppa, Berman-Berndtsson; with previous contributions from many others).

Some existence results

When X is Fano and $L = K_X^{-1}$: the existence of a Kähler-Einstein metric is equivalent to K-stability. (Chen, Donaldson, Sun and subsequent alternative proofs by Szekelyhidi, Chen-Sun-Wang, Berman-Boucksom-Jonsson.)

Examples are known of K-stable and K-unstable Fano manifolds. It is an interesting question whether the moduli spaces of stable bundles on curves $M(r,d)$ for $(r,d)=1$ are K-stable.

In the Fano case one can invoke many deep results about Riemannian manifolds with Ricci curvature bounds and from the theory of the complex Monge-Ampère equation.

Little was known about the general CSC case until recently, with work of Chen and Cheng. They proved that control of the scalar curvature and the "entropy" gives control of a Kahler metric. The entropy is

$$\int V \log V,$$

where V is the volume element, relative to a fixed reference form.

The entropy appears as the highest order term in the Mabuchi functional, related to the $-\log\det\nabla^2 u$ term in (3). This gives existence results in situations where the remaining terms can be controlled.

Examples (Chen and Cheng)

- A polarised toric manifold admits a CSC metric isf and only if it is “uniformly K-stable, i.e.

$$L(f) \geq \epsilon \|f\|,$$

for some $\epsilon > 0$ and for all non-trivial rational PL convex f. (When $n = 2$ one can remove “uniformly”.)

- A surface with $c_1 < 0$ and no curves of negative self-intersection admits a CSC metric in any Kähler class. (On the other hand, there are examples of products of curves which are not K-stable for some polarisation (Ross).)

Chen and Cheng also proved that “analytic” stability, involving geodesic rays in $\mathcal{H}$, is equivalent to the existence of a CSC metric.