

Motives of moduli spaces of bundles on curves

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Joint work with Lie Fu & Simon Pepin Lehalleur

§ 1 Overview

Let C/k be a smooth projective curve

Fix: rank n & degree d

Various moduli spaces:

moduli space of
semistable
vector bundles

(resp. with fixed)
determinant L

projective variety of dim. $n^2(g-1)+1$
smooth if $(n,d)=1$

$$N = N_C(n, d)$$

$$\text{(resp. } N_L \text{)}$$



moduli space of
semistable
Higgs bundles

$$M = M_C(n, d)$$

$$\text{(resp. } M_L \text{)}$$

quasi-proj. dim $M = 2 \dim N$ ($T^*N \cap M$)
variety smooth if $(n, d) = 1$

moduli space of
 α -semistable
parabolic bundles

$$D = \{p_1, \dots, p_N\}$$

$$N^\alpha = N_{G,D}^\alpha(n, d, m)$$

parabolic
points

multiplicities
(of flags)

$$\text{(resp. } N_L^\alpha \text{)}$$

$\dim N^\alpha = \dim N + \sum_{i=1}^N \dim \mathcal{F}(m_i)$ proj. variety
smooth if α is generic

$$M^\alpha = M_{G,D}^\alpha(n, d, m)$$

m. space of
 α -semistable
parabolic Higgs bundles

$$\text{(resp. } M_L^\alpha \text{)}$$

quasi-proj. dim $M^\alpha = 2 \dim N^\alpha$
variety smooth if α is generic

Goal: Describe the motives of these moduli spaces

motive: in the sense of Grothendieck (via Chow motives)

* encodes cohomology groups:

- ($k = \mathbb{C}$) singular cohomology + mixed Hodge structure
- ℓ -adic cohomology + Galois representation

* and algebraic cycles (Chow groups)

§2 Chow motives

Effective Chow motives

$$\begin{array}{ccccc}
 \text{SmProj}(k) & \longrightarrow & \text{Corr}(k, \mathbb{Q}) & \xrightarrow{\substack{\text{idempotent} \\ \text{completion}}} & \text{CHM}^{\text{eff}}(k, \mathbb{Q}) \\
 \text{ob: } X \text{ sm. proj.} & & X & & (X, p) \xrightarrow{p \in \text{CH}^{d_X}(X \times X)_\mathbb{Q}} \text{CH}^{d_X}(X \times X)_\mathbb{Q} \\
 & & & & \text{idempotent } (p \circ p = p)
 \end{array}$$

hom: $f: X \rightarrow Y$ $[f] \in \text{Hom}(X, Y) := \text{CH}^{d_Y}(X \times Y)_\mathbb{Q}$ $\text{Hom}((X, p), (Y, q)) := q \circ \text{CH}^{d_Y}(X \times Y)_\mathbb{Q} \circ p$

$$\begin{aligned}
 h: \text{SmProj}(k) &\longrightarrow \text{CHM}^{\text{eff}}(k, \mathbb{Q}) \quad \text{symmetric monoidal functor} \\
 X &\mapsto h(X) := (X, \Delta_X) \qquad \qquad \hookrightarrow h(X \times Y) = h(X) \otimes h(Y) \\
 \text{Spec } k &\mapsto h(k) = (\mathbb{Q}(0)) \quad \text{unit for } \otimes \\
 \mathbb{P}^1 &\mapsto h(\mathbb{P}^1) = \mathbb{Q}(0) \oplus \mathbb{Q}(1) \xleftarrow{\text{Tate twist}} \\
 \mathbb{P}^n &\mapsto h(\mathbb{P}^n) = \bigoplus_{i=0}^n \mathbb{Q}(i)
 \end{aligned}$$

$$\text{Chow motives: } \text{CHM}^{\text{eff}}(k, \mathbb{Q}) \hookrightarrow \text{CHM}(k, \mathbb{Q}) \xleftarrow{\text{⊗-invert } \mathbb{Q}(1)}$$

Voevodsky's embedding thm:

$$\begin{array}{ccc}
 \text{CHM}^{\text{eff}}(k, \mathbb{Q}) & \hookrightarrow & \text{DM}^{\text{eff}}(k, \mathbb{Q}) \\
 \uparrow h & & \uparrow M \\
 \text{SmProj}(k) & \hookrightarrow & \text{Var}(k)
 \end{array}$$

Properties

- Universal property: any Weil cohomology on $\text{SmProj}(k)$ factors via h
- Chow groups: $X \in \text{SmProj}(k)$: $\text{CH}^i(X)_\mathbb{Q} \simeq \text{Hom}_{\text{CHM}}(h(X), \mathbb{Q}(i))$
- Projective bundle formula, blow-up formula...

§3 Results on $h(N)$ ↑ moduli space of semistable vector bundles on C

Thm A [Fu-H-Pepin Lehalleur] Assume $(n,d)=1$

i) $h(N_L) \otimes h(N)$ lie in the tensor subcat. $\mathcal{C} = \langle h(C) \rangle^\otimes \subset \text{CHM}(k, \mathbb{Q})$

ii) $h(N) \cong h(N_L) \otimes h(\text{Jac}(C)) \in \text{CHM}(k, \mathbb{Q})$

i) adapts an argument of Beauville & Bülls via Chern classes of the univ. family

ii) refines the isomorphism of Harder-Narasimhan on ℓ -adic cohomology

Pf of ii) Recall $\Gamma_n = \text{Jac}(C)[n] \cap N_L$ via $M \cdot E = E \otimes M^{-1}$

and there is an isomorphism

$$N_L \times^{\Gamma_n} \text{Jac}(C) \xrightarrow{\sim} N$$

Consequently $h(N) \cong (h(N_L) \otimes h(\text{Jac}(C)))^{\Gamma_n}$

We show the Γ_n -action on $h(N_L)$ & $h(\text{Jac}(C))$ is trivial:

- (1) Reduce to a field k of char 0.
 - (2) $h(N_L)$ is abelian by i).
 - (3) In char 0, ℓ -adic realisation is conservative on abelian geom. motives & $\Gamma_n \curvearrowright H^*(N_L, \mathbb{Q}_\ell)$ is trivial.
- [Harder-Narasimhan] □

Thm B [Fu-H.-Pepin Lehalleur]

- i) Positive formulae for N_L in the \mathbb{L} -localised Grothendieck group of Chow motives lift to $\text{DM}(k, \mathbb{Q})$. \rightsquigarrow suffices to eliminate negative signs in HN recursion
- ii) Formulae for Chow motive in low ranks: $n=2$ and $n=3$ & coprime d

$$h(N_L(2,d)) \cong h(\text{Sym}^{g-1}(C))(g-1) \oplus \bigoplus_{i=0}^{g-2} h(C^{(i)}) \otimes [\mathbb{Q}(i) \oplus \mathbb{Q}(3g-3-2i)]$$

$$h(N_L(3,d)) \cong \bigoplus_{\substack{i,j \geq 0 \\ i+j \leq 2g-2}} h(C^{(i)}) \otimes h(C^{(j)}) \otimes T_{i,j}$$

$\overbrace{\hspace{10em}}$
explicit sum of pure Tate twists

- i) uses Thm A i) & a conservativity argument
- ii) relies on work of Thaddeus & del Baño ($n=2$) & Gomez-Lee ($n=3$)

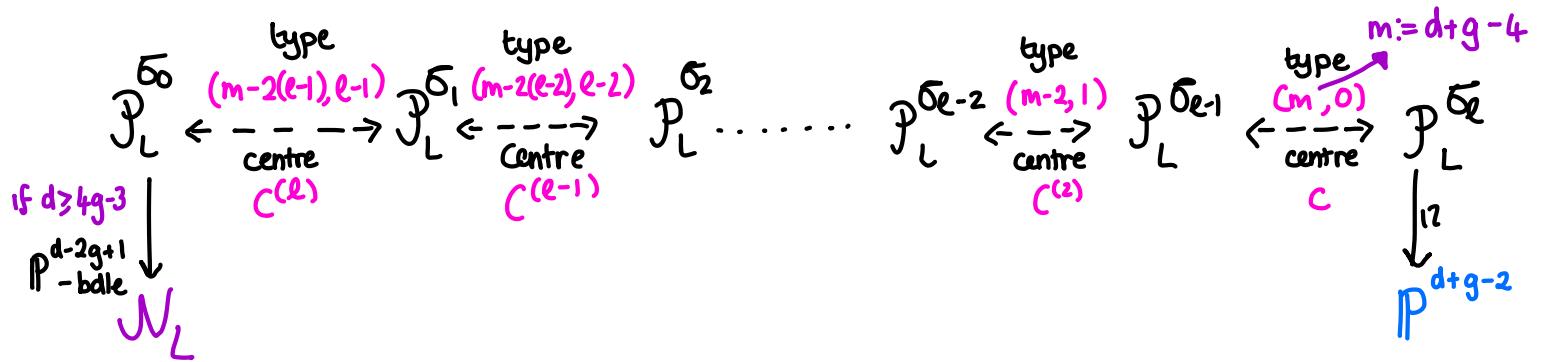
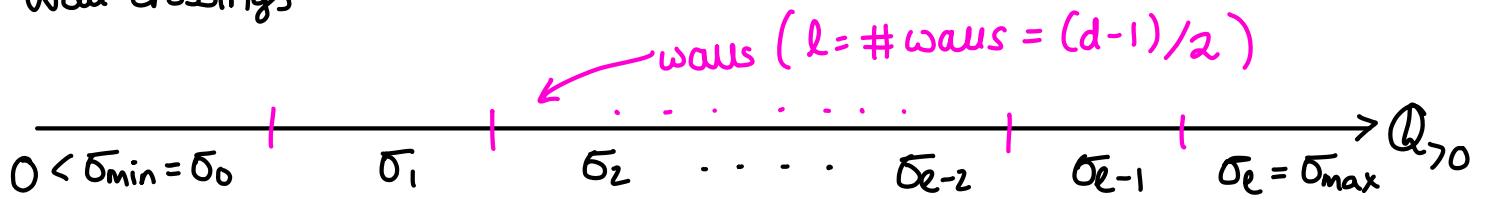
Sketch of ii) for $n=2$: WLOG (by tensoring with a line bundle): $d > 0$

We use rank 2 pairs [Bradlow] & wall-crossings [Thaddeus]

For $\sigma \in \mathbb{Q}_{>0}$ \exists proj. moduli spaces $P_L^\sigma(2, d)$ of σ -ss pairs (E, ϕ)
 stability param. \hookrightarrow smooth if $\sigma\text{-ss} = \sigma\text{-s}$

$r_k = 2$ }
 $\det = L$ } \hookleftarrow non-zero section

Wall-crossings



Def: A birat^e map of smooth proj. varieties $X \dashrightarrow X'$ is a standard flip of type (a, b) with smooth proj. centre S if $\exists Z \dashrightarrow X$ & $Z' \dashrightarrow X'$ projective bundles over S s.t.:

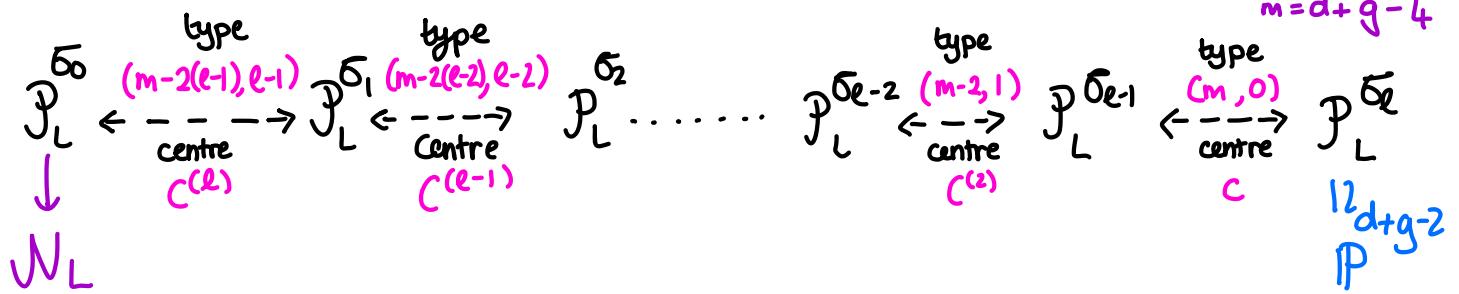
$$\begin{array}{ccc} Bl_Z(X) \cong Bl_{Z'}(X') & & \text{notation:} \\ \searrow & \swarrow & \\ Z \dashrightarrow X \dashrightarrow X' \dashrightarrow Z' & & X \xleftarrow[\substack{\text{type} \\ (a, b)}]{\substack{\text{centre} \\ S}} X' \\ \searrow \mathbb{P}^a\text{-bdle} & & \swarrow \mathbb{P}^b\text{-bdle} \end{array}$$

"flop" if $a=b$

In $\hat{K}_0(\text{CHM}(k, \mathbb{Q}))$: $\chi(x) = \chi(x') + \chi(S) \cdot (\chi(\mathbb{P}^a) - \chi(\mathbb{P}^b))$

Thm [Jiang] If $a \geq b$ $h(x) \simeq h(x') \oplus \bigoplus_{j=b+1}^a h(S)(j)$ in $\text{CHM}(k, \mathbb{Z})$

Working from right to left in the pairs wall-crossing diagram:



- Initially we have flips of type (a, b) with $a > b$
→ good news: h increases

$$h(P_L^{\delta_{e-j}}) = h(P_L^{\delta_e}) \oplus \text{contributions from the centres of each flip}$$

$$\hookrightarrow \text{for small } j \quad (j \leq \frac{d+g-1}{3}) \quad \text{in } P_L^{\delta_{e-j}} \xleftarrow[a \geq b]{\text{type } (m-2(j-1), j-1)} P_L^{\delta_{e-j+1}} \quad m = d + g - 4$$

- However for larger j we get flips of type (a, b) with $a \leq b$
→ bad news: we need to "cancel" part of the motive

Solution: • work in the (lI-completed) Grothendieck group $\hat{K}_0(\text{CHM}(k, \mathbb{Q}))$

[del Bño]: Computation in $\hat{K}_0(\text{CHM}(k, \mathbb{Q}))$

- eliminate minus signs: find a positive expression for $X(N_L)$ and apply Thm B i). □

§4 Results on $h(M)$

moduli space of
ss Higgs bundles Δ quasi-proj.
variety

Thm C [H.-Pepin Lehalleur] Assume $(n, d) = 1$

- The motive of M is pure. Thus $h(M) \in \text{CHM}(k, \mathbb{Q})$
- Assume $C(k) \neq \emptyset$. Then $h(M) \in C = \langle h(C) \rangle^\otimes$.

The starting point for the proof is the Bialynicki-Birula decomposition of M associated to Hitchin's scaling action $G_m \curvearrowright M$ $t \cdot [E, \Phi] = [E, t\Phi]$

- fixed locus is projective

$$M^{G_m} = N \amalg \begin{array}{l} \text{moduli spaces} \\ \text{of chains} \end{array} \rightsquigarrow E = \bigoplus E_i \xrightarrow{\Phi} E_1 \otimes w_1 \xrightarrow{\Phi^2} E_2 \otimes w_2 \xrightarrow{\Phi^3} \dots \quad \left. \begin{array}{l} \text{"semi-proj."} \\ G_m\text{-action} \\ [\text{Hausel}] \end{array} \right\}$$

- the flow as $t \rightarrow 0$ exists $\forall [E, \Phi] \in M$

[García-Prada - Heinloth - Schmitt] study this decomposition & the classes of chain moduli spaces in the Grothendieck ring of varieties via variation of stability.

Rmk [Fu - H. - Pepin Lehalleur] The $S_{\mathbb{L}}$ -Higgs moduli space M_L for a general curve C/\mathbb{C} has motive $h(M_L) \notin \langle h(C) \rangle^{\otimes}$.
 In particular $h(M) \not\cong h(M_L) \otimes h(\text{Jac}(C))$

Already seen in rank $n=2$ [Hitchin] ↪

$\Gamma_n \cap h(M_L)$
non-trivially

Thm D [Fu - H. - Pepin Lehalleur] (Formulae for $n=2$ & 3 & coprime d)

$$h(M(2,d)) \simeq h(N(2,d)) \oplus \bigoplus_{j=1}^{g-1} h(\text{Jac}(C)) \otimes h(C^{(j-1)}) (3g-2j-2)$$

holds with \mathbb{Z} -coeffs

$$h(M(3,d)) \simeq h(N(3,d)) \oplus \bigoplus_{i \in I} h(J_C) \otimes h(C^{(i)}) \otimes T_i \oplus \bigoplus_{(i,j) \in J} h(J_C) \otimes h(C^{(i)} \times C^{(j)}) \otimes T_{i,j}$$

with \mathbb{Q} -coeffs

↑
Jac(C)

↑
 $i \in I$

↑
 $(i,j) \in J$

→
explicit sums of Tate twists

Generalises cohomological results of Hitchin ($n=2$) and Gothen ($n=3$).

Sketch of the proof: We study the BB decomp. associated to

$$\begin{aligned} \mathbb{G}_m \curvearrowright M &\rightsquigarrow M = N \coprod_{(\underline{m}, \underline{e})} \text{Ch}_{\underline{m}, \underline{e}}^{\alpha_H - ss} \\ \text{Hitchin's scaling action} \end{aligned}$$

α_H = Higgs stability param. for chains

tuple of ranks & degrees

Motivic BB decomp. (with \mathbb{Z} -coeffs)

$$h(M) = h(N) \oplus \bigoplus_{(\underline{m}, \underline{e})} h(\text{Ch}_{\underline{m}, \underline{e}}^{\alpha_H - ss}) (\text{codim}_{\underline{m}, \underline{e}})$$

In low ranks we can analyse the types $(\underline{m}, \underline{e})$ of chains appearing & describe the motives of the corresponding chain moduli spaces.

Rank 2 • $\underline{m} = (2)$ & $\underline{e} = (d)$ $\rightsquigarrow \mathcal{N}(2, d)$

(Hitchin) • $\underline{m} = (1, 1)$ & $\underline{e} = (e_1, e_2)$ $e_2 = d - e_1 + 2g - 2$

stability & $\phi \neq 0$ $L_0 \xrightarrow{\phi} L_1 \otimes \omega_C$ \rightsquigarrow param. by $\text{Pic}^{e_1}(C) \times C^{(e_2 - e_1)}$

\Rightarrow finitely many possible values of e_1

Tate twist = codim of BB stratum
(calculate using fact that downward flow is Lagrangian)

Rank 3 • $\underline{m} = (3)$ & $\underline{e} = (d)$ $\rightsquigarrow \mathcal{N}(3, d)$

(Gothen) • $\underline{m} = (1, 1, 1)$ & $\underline{e} = (e_1, e_2, e_3)$ $e_1 + e_2 + e_3 = d + 6(g-1)$

$$L_0 \rightarrow L_1 \otimes \omega_C \rightarrow L_2 \otimes \omega_C^{\otimes 2}$$

\rightsquigarrow param. by $\text{Pic}^{e_0}(C) \times C^{(e_2 - e_1)} \times C^{(e_3 - e_2)}$

finitely many possibilities for \underline{e} in each case

• $\underline{m} = (1, 2)$ & $\underline{e} = (e_1, e_2)$

$L \rightarrow E \otimes \omega_C \rightsquigarrow$ rk 2 pair ($F = E \otimes \omega_C \otimes L^{-1}, \phi$)
section

\rightsquigarrow param. by $\text{Pic}^{e_1}(C) \times \mathbb{P}^{\sigma\text{-ss}}(2, f)$ ← moduli space of σ -semistable pairs of rk 2 degree f

Motives computed by pairs wall-crossings (explicit flips)
as in the proof of Thm B.

• $\underline{m} = (2, 1)$ & $\underline{e} = (e_1, e_2)$ (dual to $\underline{m} = (1, 2)$)

$F \Rightarrow L \otimes \omega_C \rightsquigarrow$ param by $\text{Pic}^{e_2}(C) \times \mathbb{P}^{\sigma\text{-ss}}(2, f)$ \blacksquare

Rmk: The BB decomposition for $\mathcal{M}_L(2, d)$ gives

$$\mathcal{M}_L(2, d) = \mathcal{N}_L(2, d) \amalg \coprod_{j=1}^{g-1} \widetilde{C}^{(2j-1)} \quad [\text{Hitchin}]$$

where $\widetilde{C}^{(j)} \xrightarrow[\text{étale cover}]{\deg 2^{2j}} C^{(j)}$ $\rightsquigarrow h(\mathcal{M}_L(2, d)) \in \langle h(\widetilde{C}) \rangle^\otimes$

$\downarrow \qquad \qquad \downarrow$

$\text{Jac}(C) \xrightarrow{\cdot 2} \text{Jac}(C)$

For a general complex curve C :

$$\langle h(C) \rangle^\otimes \subsetneq \langle h(\widetilde{C}) \rangle^\otimes$$

§5 Results on $\mathcal{H}(N^\alpha)$ and $\mathcal{H}(M^\alpha)$

moduli spaces of α -ss
parabolic (Higgs) bundles

Fix $D = \{p_1, \dots, p_N\}$ parabolic points on C .

Def: A (quasi)-parabolic bundle on (C, D) is $E_* = (E, E_{i,j})$

vector bdl

flag in E_i

$$E_{pi} = E_{i,1} \supseteq E_{i,2} \supseteq \dots \supseteq E_{i,e_i} \supseteq E_{i,e_i+1} = 0 \quad \forall p_i \in D$$

e_i = length of the flag

$$m_{i,j} := \dim(E_{i,j}/E_{i,j+1}) > 0 \quad \text{flag multiplicities}$$

• A (quasi)-parabolic Higgs bundle on (C, D) is $(E_*, \Phi: E \rightarrow E \otimes \omega_C(D))$

(quasi)-parabolic
vector bundle

strongly parabolic
Higgs field

$$\hookrightarrow \text{ie } \Phi(E_{i,j}) \subset E_{i,j+1} \otimes \omega_C(D)$$

$\epsilon R^{>0}$

Def: E_* is (semi)stable w.r.t. $\alpha = (\alpha_{i,j})$ if $\forall E' \subset E$

$$\mu_\alpha(E') \leq \mu_\alpha(E) = \frac{\deg(E) + \sum_{p_i \in D} \sum_{j=1}^{e_i} \alpha_{i,j} m_{i,j}}{\text{rk}(E)}$$

$$m'_{i,j} := \dim(E'_{pi} \cap E_{i,j}/E'_{pi} \cap E_{i,j+1}) > 0$$

Moduli spaces: [Metha - Seshadri, Yokogawa]

$N_{G,D}^\alpha(n, d, \underline{m})$ is the moduli space of α -ss parabolic vector bundles of rank n , degree d with multiplicities $\underline{m} = (m_{i,j})$.

$$\dim N_{G,D}^\alpha(n, d, \underline{m}) = \underbrace{n^2(g-1) + 1}_{\dim(N_G(n, d))} + \sum_{p_i \in D} \sum_{j > k}^{e_i} m_{i,j} m_{i,k}$$

dim flag variety $\mathcal{F}\ell(\underline{m})$

$M_{G,D}^\alpha(n, d, \underline{m})$ is the moduli space of α -ss parabolic Higgs bundles of rank n , degree d with multiplicities $\underline{m} = (m_{i,j})$

$$\exists \mathbb{G}_m \cap M^\alpha \xrightarrow{\text{semi-projective}} \mathbb{G}_m \Rightarrow \mathcal{H}(M^\alpha) \in \text{CHM}(k, \mathbb{Q})$$

motivic
BB decomp.

$$\dim(M^\alpha) = 2 \dim(N^\alpha)$$

Variation of stability for parabolic vector bundles

[Boden-Hu, Boden-Yokogawa, Thaddeus]

Fix invariant $\eta = (n, d, m)$

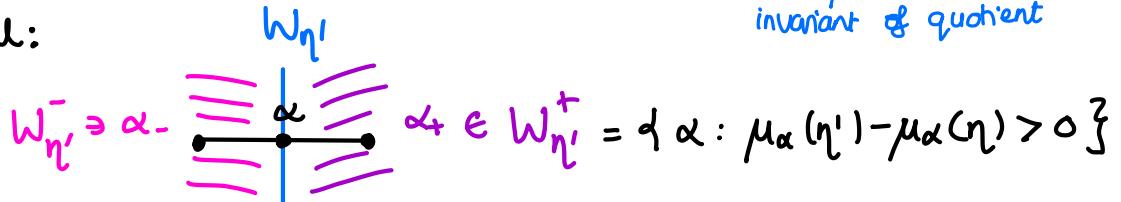
\exists walls $W_{\eta'}$ $\subset \mathcal{G} = \{(\alpha_i, j)\}_{i=1}^N \cong \mathbb{R}^N$

invariant of
Subbundle

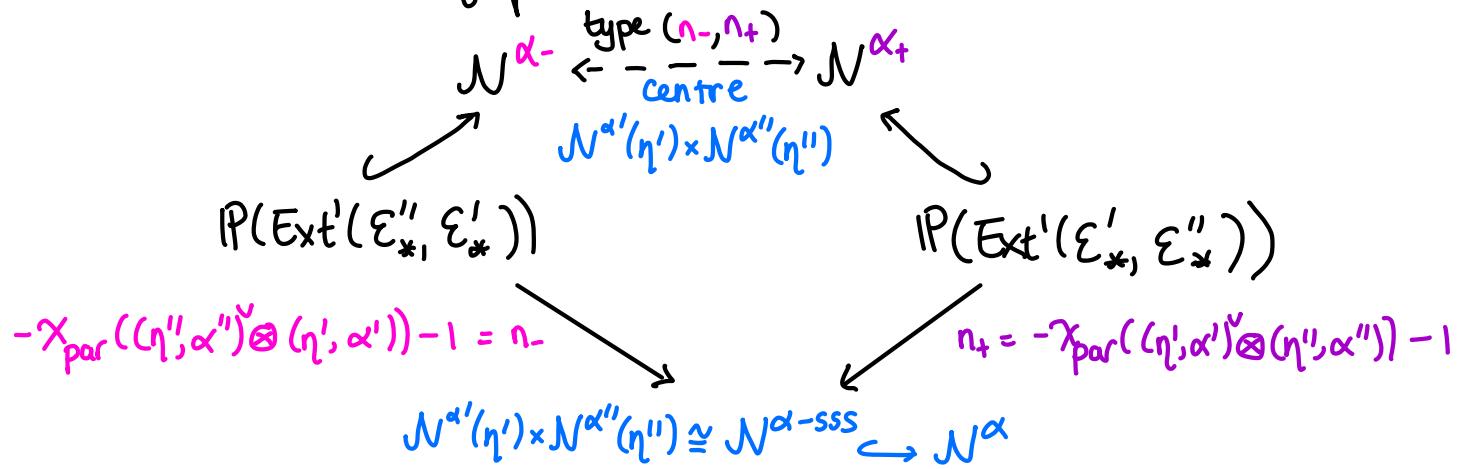
" $\{\alpha : \mu_\alpha(\eta') = \mu_\alpha(\eta)\}$ "

Assume $m = 1$ (full flags)

At a single wall:



there is a standard flip



Cor [Fu-H.-Pepin lehalleur]

$$\text{i) } h(N^\alpha) \oplus \bigoplus_{n_+ < j \leq n_-} h(N^{\alpha-\text{sss}})(j) \cong h(N^{\alpha+}) \oplus \bigoplus_{n_- < j \leq n_+} h(N^{\alpha-\text{sss}})(j)$$

\nwarrow at least one of \nearrow
these sums is empty

ii) For $i \in \mathbb{N}$, provided $g \gg 0$: $\mathrm{CH}^i(N^\alpha) \cong \mathrm{CH}^i(N^\beta)$ for α, β generic
(depending only on η & i) & same for CH_i

Rmk (flag degenerations) [Boden-Yokogawa] Assume $(n, d) = 1$

For suff. small (generic) α : α -ss of $E_* = (E, E_{i,j}) \Leftrightarrow$ ss of E .

Thus there is a forgetful map $N^\alpha \rightarrow N$ which is a flag bundle.

Consequently: $h(N^\alpha) = h(N) \otimes \bigoplus_{p \in D} h(\mathcal{F}(m_p))$ $\xrightarrow{\text{flag var.}}$ $h(\mathcal{F})$ direct sum of Tate twists
for α suff. small

Explicit formulae in rank 2: N parabolic pts $D = \{p_1, \dots, p_N\}$ with full flags.

WLOG $\begin{cases} \alpha_{i,1} = 0 \forall i \\ \alpha_{i,2} \in (0,1) \end{cases}$ (notion of ss preserved by certain shifts)

Space of weights: $\mathbb{A}_N = (0,1)^N$

Symmetries: ① Hecke modifications at $D'CD$

$$H_{D'}: N^\alpha(\eta) \xrightarrow{\sim} N^{K_{D'}}(2, d - 1D'1, m)$$

$$\text{If } |D'| = 2e: \mathcal{O}(e) \otimes H_{D'}: N^\alpha(\eta) \xrightarrow{\sim} N^{K_{D'}}(\eta)$$

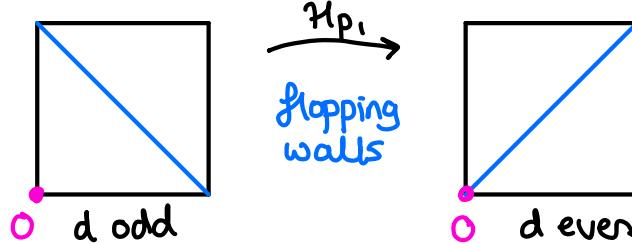
Hecke modifications at pairs $(p_i, p_{i+1}) \rightsquigarrow (\mathbb{Z}/2\mathbb{Z})^{N-1} \cap \mathbb{A}_N$

② $S_N \cap \mathbb{A}_N \rightsquigarrow$ does not give isomorphic moduli spaces

However: preserves type (n_-, n_+) of flip at wall-crossings

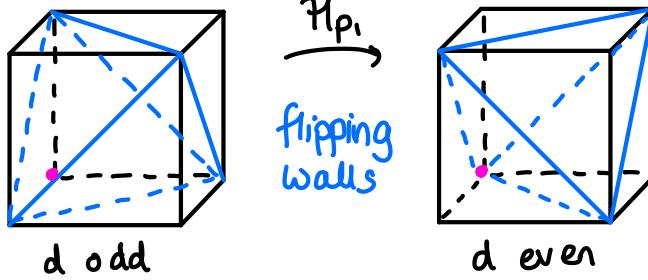
Pictures of wall-crossings for low N :

• $N=2$



$$h(N^\alpha) = h(N) \otimes h(\mathbb{P}')^{\otimes 2}$$

• $N=3$



outer chambers are permuted by Hecke modifications at pairs of parabolic points $\{p_i, p_j\}$

$$\alpha_{\text{ext}} \in \text{Ext} \text{ of tetrahedron}: h(N^{\alpha_{\text{ext}}}) = h(N) \otimes h(\mathbb{P}')^{\otimes 3}$$

$$\alpha_{\text{in}} \in \text{Int} \text{ of tetrahedron}: h(N^{\alpha_{\text{in}}}) = h(N^{\alpha_{\text{ext}}}) \oplus h(\text{Jac}(C))^{\otimes 2}(q)$$

In general: i) moving from the exterior to the centre of $\mathbb{A}_N = (0,1)^N$ increases h
ii) A wall is a flopping wall \Leftrightarrow it contains the centre. Only happens if N is even

iii) We can compute $h(N^\alpha)$ for even d via Hecke modifications

Thm E [Fu-H.-Pepin Lehalleur]

For $n=2$ with full flags at N parabolic pts:

$$h(N^\alpha) \simeq h(N) \otimes h(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{j=0}^{N-3} h(\text{Jac}(C))^{\otimes 2} (g+j)^{\oplus b_j(\alpha)}$$

Holds with \mathbb{Z} -coefficients

exponents can be explicitly computed

* We know $h(N)$ with \mathbb{Q} -coeffs by Thms A & B.

Generalises cohomological results of Bauer (over \mathbb{P}^1) & Holla.

Variation of stability for parabolic Higgs bundles

Thm F [Fu - H. - Pepin Lehalleur]

Fix (C, D) and $\eta = (n, d, \perp)$. Then for a generic weight α

$$h(M_{c,D}^\alpha(\eta)) \in \text{CHM}^{\text{eff}}(k, \mathbb{Z})$$

Sketch of proof:

roughly: "alg. symplectic versions
of standard flips"

(i) [Thaddeus] M^α & M^β are related by Mukai flops

(ii) Mukai flops between smooth varieties preserve Chow groups
& preserve motives in $\text{DM}(k, \mathbb{Z})$

(Extending a result of [Lee-Lin-Wong] for Mukai flops of smooth
projective varieties)

□

Over $k=\mathbb{C}$, on the level of Betti cohomology, this result was seen in

- rank $n=2$ by Boden-Yokogawa (Nakajima showed the spaces are diffeo)
- rank $n=3$ by García-Prada - Gothen - Muñoz

Thm G [Fu - H. - Pepin Lehalleur]

For rank $n=2$ we have

$$h(M^\alpha) = h(N) \otimes h(\mathbb{P}^1)^{\otimes N} \oplus \bigoplus_{\substack{0 \leq e \leq N \\ \frac{e+1-N}{2} \leq j \leq g-1}} h(J_C) \otimes h(C^{(2j+N-e-1)})^{(3g-2j+e-2)} \oplus \binom{N}{e}$$

with \mathbb{Z} -coeffs.

References

This talk was based on joint work with KIE FU & Simon PEPIN LEHALEUR

[arXiv: 2011.14872] "Motives of moduli spaces of bundles on curves via variation of stability & flips"

[arXiv: 2102.07546] "Motives of moduli spaces of rank 3 vector bundles and Higgs bundles on a curve"

See also the references therein.