

# Quantisation of moduli spaces of meromorphic connections, and (irregular) CFT

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- 1) Nonsingular connections.
  - 2) Isomonodromy (IMD).
  - 3) Quantum IMD :  $k \geq$  and beyond.  
(w/ G. Felder).
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1) W/ out singularities. |  $\Sigma =$  closed  


orientable smooth surface,  $K = 1$  - connected  
cpt. Lie group ( $K = SU(n), n \in \mathbb{Z}_{\geq 2}$ ).

Consider iso. classes of flat K-connections

on  $\Sigma$ :  $M_f = M_{f_1}(\Sigma, K)$ , a

(singular) symplectic space [Atiyah-Bott, 1982].

Identified w/ K-character variety:

$\text{Hom}(\pi_1(\Sigma), K)/K$ , taking monodromy data.

[Goldmann, 1983]

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• Quantisation? Choose a

complex / conformal structure on  $\Sigma$ :

it's now a Riemann surface.

Thm .] [Narasimhan - Seshadri, 1965]

$M_{\text{fl}}^{\text{irr}} \subseteq M_{\text{fl}}$  identified w/ mod.

space  $\overset{\text{St}}{\underset{G}{\text{Bun}}} = \text{Bun}_G^{\text{st}}(\Sigma)$  of iso. classes  
of stable top. trivial holomorphic  
principal  $G$ -bundles on  $\Sigma$ , for  
a complexification  $K \hookrightarrow G$   
(think  $G = \text{SL}_n(\mathbb{C})$ ).

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Hence we have a Kähler manifold,  
can do geometric quantisation. Take  
the determinant bundle [Quillen, 1985]  
 $L \rightarrow \overset{\text{St}}{\underset{G}{\text{Bun}}}$ , and for  $\kappa = 1/\hbar \in \mathbb{Z}_{>0}$ :

$$\mathcal{H}^{(n)} = H^0(Bun_G^{\text{st}}, L^{\otimes k}).$$

Depends on a parameter in  
 the Riemann mod. space  $B := \mathcal{M}_g$   
 $\delta = \text{genus } (\Sigma)$ ; or in Teichmüller  
 space  $T_g \rightarrow B$  (in  $MCG$ -equiv.  
 way).

Now recall the "symplectic nature" of  
 $\pi_1(\Sigma)$ : the mod. spaces assemble into  
 a local system  $\widetilde{Bun}_G \rightarrow B$  of  
 symplectic manifolds (i.e. a symplectic  
 fibre bundle w/ flat complete symplectic  
 Chernmann connection).

On the quantum side : we want a  
(projectively) flat vector bundle

$$\tilde{\mathcal{H}}^{(n)} \rightarrow B.$$

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Theorem. [Hitchin, Axelrod-Della Pietra -  
- Witten, 1990]

Construction of (proj.) flat connection.

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Summary

$$(\Sigma, K)$$

↓  
Classical (T)FT

[Symplectic local system over]

$B =$  space of deformations of

[R. surface structures on  $\Sigma$ ]

$( + \pi_1(B) - \text{action} ) .$

Analogously :

$(\mathcal{E}, K, \kappa)$

$\begin{cases} \text{(T)QFT} \\ \downarrow \end{cases}$

$\lceil \text{Projectively flat vector bundle} \rceil$

$\Downarrow \text{over } B \Downarrow$

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$( + \text{quantum } \pi_1(B) - \text{action} ) .$

① Other approaches for quantisation : CFT,

quantum groups.

② Applications : invariants of links in  
3-manifolds via quantum partition

functions (recovers Jones polynomials [Witten, 1989]; see Atiyah's book "Geometry and Physics of Knots").

Extension | Start from  $G$  cplx.  
reductive (i.e. cplx. Chern-Simons).

Then consider iso. classes of flat  $G$ -connections on  $\Sigma$ :

$$\mathcal{M}_{\text{dR}} = \mathcal{M}_{\text{dR}}(\Sigma, G)$$



$$\mathcal{M}_B = \mathcal{M}_B(\Sigma, G)$$

$$= \text{Hom}(\pi_1(\Sigma), G)/G.$$

Now Atiyah-Bott/Goldmann form is

holomorphic. If  $\Sigma$  is Riemann

then  $M_{dR} \simeq (M_H, J)$ , where

$M_H = M_H(\Sigma, G)$  is the mod. space  
of self-duality/Hitchin eq's.

It is hyperkähler (cf. Alessandro's talk).

Further  $(M_H, I) \simeq M_{Dol} = M_{Dol}(\Sigma, G)$ ,

mod. space of  $G$ -Higgs bundles on

$\Sigma$ : by hyperkähler rotation  $M_{dR} \simeq M_{Dol}$ ,

the nonabelian Hodge correspondence

[Hitchin, Donaldson, Corlette, Simpson, 1987-1994].

Still true:  $\hat{\mathcal{M}}_{\text{dR}} \rightarrow \mathbb{B} = \mathcal{M}_y$  is a

holomorphic symplectic local system.

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② Quantisation?

Algebraically: if  $A_0 = C^0(\mathcal{M}_{\text{dR}}, \mathbb{C})$ ,

deformation-quantise to get a vector  
bundle  $\tilde{A} \rightarrow \mathbb{B}$  of quantum  $\mathbb{C}[\hbar]$ -algebras:  
converges a (proj.) flat connection.

(See later more generally).

① Geometrically: NAK hyperKähler structure

yields  $\mathbb{CP}^1$ -bundle  $\mathbb{B}' \rightarrow \mathbb{B}$ . Hence

a fibration  $\tilde{\mathcal{M}}_{\text{dR}} \rightarrow \mathbb{B}'$  of Kähler

manifolds.

Now we can: fix  $\varepsilon \in M_g$ , let  $q \in \mathbb{P}^1$  vary (cf. yesterday); fix  $q \in \mathbb{P}^1$ , let  $\varepsilon \in M_g$  vary; vary both.

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Question: in which generality do we

have sympl. local systems over  $B^1$ ?

And (prob.) flat connection after Kähler quantisation?

[Note] Not all will be prequantisable: use a level  $b = k + i\beta \in \mathbb{C}^\times$ .

[Witten, 1981; Andersen - Malusai - R.,  
arXiv: 2012.15630 ]

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2) Eisenhart deformations.

$\} \text{flat } G\text{-conn. on } \Sigma \text{ smooth} \} /_{(\text{id.})}$

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$\} \text{holomorphic } G\text{-conn. on } \Sigma \text{ Riemann} \} /_{(\text{id.})}$

12 (GAGA)

$\} \text{non sing. algebraic } G\text{-conn. on}$

$\Sigma \text{ projective curve } (C) /_{(\text{id.})}$

We extend the latter: consider alge.

connections w./ (possibly) irregular  
 $\sim$

singularities, i.e. meromorphic w./  
high-order poles.

Locally @ pole we have  $\nabla = d - A$ ,

$$A = (\lambda_2 dz + d\lambda_2) + g \bar{e}_2 \bar{J} dz,$$

$g = \text{Lie}(G)$ ,  $\lambda \in g$  and

$$(k) \quad \lambda = \sum_{i=1}^p T_i / z^i, \quad T_i \in g.$$

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Def] An untwisted irregular type

@  $\alpha \in \Sigma$  is an element

$$Q \in V((z)) / V[\bar{e}_2] \simeq z^{-1} V[z^{-1}],$$

for  $V \subseteq g$  Cartan and  $f(\alpha) = 0$

(i.e. (k) w/  $T_i \in V$ ).

E.g. if  $T_p \in V_{reg} \subseteq V$  then

$\mathbf{E}$  is untwisted up to formal gauge (generic case).

Consider now  $\underline{\alpha} := \{ \alpha_1 - \alpha_m \} \subseteq \Sigma$  (ordered) marked points,  $\underline{\beta} = \{ \beta_1 - \beta_{2m} \}$  untwisted irr. types  $@ \underline{\alpha} \subseteq \Sigma$ .

Facts | @  $M_{dR} = M_{dR}(\Sigma, G)$ , mod.

# Space of irregular $G$ -connections

oh  $\sum$  w./ poles @  $a \subseteq \Sigma$  and

principal parts controlled by  $\underline{\Omega}$ :

i't') Poisson.

Here  $\underline{\Sigma} = (\Sigma, \underline{\alpha}, \underline{\mathbb{Q}})$  is a wild Riemann surface [Boalch, 2014].



(Fixing "residue orbits"  $\underline{\partial} = \{\partial_+, -, \partial_m\}$  we get symplectic spaces).

- $M_{dR} \simeq M_B = M_B(\underline{\Sigma}, G)$ , wild character variety: Stokes data, more than  $\widehat{\mathrm{H}}_1(\Sigma)$ -modules (also taking monodromy data).
- If  $\underline{\Sigma}$  is deformed in a space  $B$ , get holomorphic symplectic local systems  $\tilde{M}_{dR} \rightarrow B$ . The flat

connection is the pull-back of the natural one on  $\tilde{M}_B \rightarrow B$ : the IMD connection.



[Boalch, Boalch - Yamakawa, 2001-2015].

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Summary |  $(\Sigma, G)$



[Holomorphic symplectic  
local system over

$B =$  space of deformations

of wild R. surface structures

on  $\Sigma$

(+  $\pi_*(B)$ -action;  $B$  generalises  $M_{g,m}$ ).

Main problem

Quantise this!

E.g.  $\nabla = d - A$ ,  $A = \sum_{i=1}^m \frac{a_i}{z - b_i} dz$ , on

$$\Sigma = \mathbb{C}P^1 \text{ (i.e. } \underline{\Omega} = 0\text{).}$$

The IMD connection is equivalent  
to Schlesinger system (1812):

$$H_i = \sum_{j \neq i} \frac{\text{Tr}(A; A_{ij})}{t_i - t_j},$$

$$H : \mathcal{G}^n \times B \rightarrow \mathbb{C}^m, B = \text{Conf}_m(\mathbb{C}).$$

(Generalises PVI; consider trivial bundles).

①  $A = (T + \lambda/z) dz$ , pole of

order 2 at  $\infty$  ( $Q_{\infty} = T/w$ ,

$w = z^{-1}$ ),  $T \in \mathbb{R}_{\text{reg}}$ .

IMD connection given by

dual system [Harnad, 1984].

②  $A = (T + \sum_{i=1}^m \lambda_i/z - t_i) dz$ .

IMD connection given by system of

Jimbo-Miwa-Môri-Sato [JMS, 1980].

③  $A = (A\bar{z} + B + \sum_{i=1}^m \lambda_i/z - t_i) dz$ .

IMD connection given by  
 Simply-laced IMD systems (SLIMS)  
 [Boalch, 2012].

- ① U w.l.o. arbitrary poles : System of Klavé's (1979). But only along variations of marked points.

IMD system	Poles	$B$
Schlesinger	$m \times 1$	$M_{0,n}$
dual Schlesinger	$2 + m$	$V_{reg}$
JMMS	$2 + m \times 1$	$M_{0,n} \times V_{reg}$
SLIMS	$3 + m \times 1$	$\prod_i \text{Conf}_{m_i}$

## 3) Quantisation of IMD connections.

- Schlesinger  $\Rightarrow$  Khizhnik - Zamolodchikov (1984) [Reshetikhin, 1992; Harnad, 1994].
- dual Schlesinger  $\Rightarrow$  De Concini / Millson - Toledano Laredo (2001)  
[Boalch, 2002]
- JMMG  $\Rightarrow$  Felder - Markov - Tarasov - Varchenko (2000) [R., 2019]
- SLIMG  $\Rightarrow$  simply-laced quantum  
connections [R., 2019].

④ Klare's  $\Rightarrow$  irregular KZ

[Reshetikhin, 1992; Felder-R.,

arXiv. 2012: 14783]

These have all applications in  
(irregular generalisations of) the  
WZNW model.

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CFT ideas | Recall about KZ.

If  $\lambda \in V^\vee$  weight, we have affine/  
finite Verma modules:

$$V_\lambda := V_\lambda \otimes_{U_f^+} C_\lambda$$

f

$$\hat{V}_{\lambda, \kappa} = U_{\hat{g}}^{\wedge} \otimes_{U_{\hat{g}}^{\wedge}} \mathbb{C}_{\lambda, \kappa}, \quad \kappa \in \mathbb{C}.$$

Here  $\hat{g} \rightarrow \hat{g} := g((z))$  is the affine Lie algebra of  $(g, (\cdot | \cdot))$ , for

$(\cdot | \cdot) : g \otimes g \rightarrow \mathbb{C}$  symm. adjoint invariant,

$\hat{b}^+ \subseteq \hat{g}$  a positive Borel

wrt. a positive root system

$$R_+ \subseteq R = R(g, \nu).$$

Note:  $\lambda \in \nu^\vee$  and  $(\lambda, \kappa) \in \hat{\nu}^\vee$  defining characters of  $b^+$  and  $\hat{b}^+$ .

Then consider  $\underline{\lambda} = (\lambda_1, \dots, \lambda_m)$ , and:

$$\mathcal{H} = \mathcal{H}_{\underline{\lambda}} := \bigotimes_{i=1}^m V_{\lambda_i}$$



$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\underline{\lambda}, K} := \bigotimes_{i=1}^m V_{\lambda_i, K}.$$

Facts  $\bullet$   $\hat{\mathcal{H}}$  carries an action of

the lie algebra of  $\mathcal{G}$ -valued

rational functions on  $\Sigma = \mathbb{CP}^1$ ,

w/ poles @  $\underline{\alpha} \subseteq \Sigma$ .

Denote it  $g(\underline{\alpha})$ .

- $\hat{\mathcal{H}}/\langle g(\alpha) \rangle \hat{\mathcal{H}} \simeq \mathcal{H}/g_{\mathcal{H}} : \text{the}$   
 space of [genus-zero WENW)  
 Verma conformal blocks.
- 
- $\overset{1}{\hat{\mathcal{H}}} \times B \rightarrow B = \text{Conf}_m(\Sigma)$  carries  
 an invariant flat connection def.  
 by  $Vir \rightarrow \mathfrak{gl}_C(\hat{U}\hat{g})$  of Segal-  
 -Sugawara: induces  $K$  on

$$\mathcal{H}/g_{\mathcal{H}} \times B \rightarrow B .$$

(See Kohno's book "CFT and topology").

Concretely:

$$\nabla^{K\ell} = d - \bar{\omega}^{K\ell}, \quad \bar{\omega}^{K\ell} = \sum_{i=1}^m \hat{H}_i dt_i,$$

$$\hat{H}_i = \frac{1}{K\hbar^{\vee}} \sum_{j \neq i} \frac{\Omega^{(ij)}}{t_i - t_j} \in \mathfrak{U}_{\mathfrak{g}}^{\otimes m}.$$

Hence  $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$  is the canonical tensor of  $(\mathfrak{g}, [\cdot, \cdot])$ .

... but  $\hat{H}_i$  is a direct quantisation of  $H_i$  from Schlesinger !

$$(\lambda_1 - \lambda_m) \mapsto \text{Tr}(\lambda_i \lambda_j)$$

↓

$$\in (\text{Sym } \mathfrak{g}^{\vee})^{\otimes m} \simeq \text{Sym } \mathfrak{g}^{\otimes m},$$

quantised by  $\mathcal{L}^{(ij)}$  using  
 the canonical symmetrisation  
 $\text{Sym } \mathfrak{g} \rightarrow \mathfrak{U}_g$ .

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Relations w/ logarithmic connections?

Idea:  $\lambda \in \mathfrak{r}^\vee$  corresponds to

$\frac{1}{z} dz \in V \otimes z^{-1} dz$  under the

pairing  $dg dz \otimes dg \rightarrow \mathbb{C}$ ,

$(X \otimes w, y \otimes f) \mapsto (X(y)) \underset{z \geq 0}{\text{Res}} (f w)$ .

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Hence  $A = \frac{1}{z} dz + dQ \in z^{-1} V[z^{-1}]$

corresponds to  $(\lambda, \varepsilon_1 \dots \varepsilon_{p-1})$

$\oplus$

$$\bigoplus_{i=0}^{p-1} (\nu \otimes \varepsilon^i)^\vee$$



$$\left( \frac{\nu \otimes \varepsilon}{\varepsilon^p \nu \otimes \varepsilon} \right)^\vee =: \nu_p^\vee.$$

$\nu_p \subseteq \mathcal{G}_p := g \otimes \mathbb{I} / \varepsilon^p g \otimes \mathbb{I}$  is the

algebra to consider.

Theorem (Felder - R., 2020)

• Construction of subalgebra  $\mathcal{G}_p \subseteq g^\vee$

whose characters are coded by

$$(\lambda, \varepsilon) \in \nu_p^\vee \text{ and } \kappa \in \mathbb{C}, \quad \mathcal{J}_\kappa = \hat{f}^+.$$

① Definition of affine (finite singularity) modules



$$W = W_{\lambda, \underline{\epsilon}}^{(p)}$$



$$\hat{W} = \hat{W}_{\lambda, \underline{\epsilon}, \kappa}^{(p)}, \quad \text{J.-L.}$$

$\hat{W}$  is naturally a  $\mathfrak{g}_p$ -module.

② Smooth modules, generated by Gaiotto - Teolini irregular state.

③  $\mathfrak{g}(\underline{a})$ -action induces generalisation

of irregular KZ on  $H/g_R \times B$

↓  
B,

$$\mathcal{H} = \bigotimes_{i=1}^m W_{\lambda_i, \underline{\epsilon}_i}^{(\rho_i)}$$

- Finite-dimensional r-weight decomposition  
of  $W$ , affine invariance,  
universal flatness of quantum  
connections on  $Ug_p^{\otimes m} \times B \rightarrow B$   
by CYBE (generalising classical  
r-Matrix  $r(t) = \mathcal{S}(t)$  of KZ),  
etc.

THANK YOU

FOR YOUR

ATTENTION /

