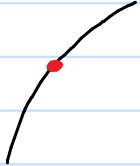


# Cohomological Hall Algebras of curves & surfaces

## Plan

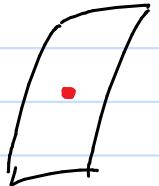
1) Point modification of coherent sheaves on curves

- \*  $\text{Fun}(\text{Coh}_X, \mathbb{C})$  for  $X/\mathbb{F}_q \rightsquigarrow$  (spherical, affine) Hecke algebra
- \*  $H_*(\text{Coh}_X, \mathbb{C})$  for  $X/\mathbb{C} \rightsquigarrow$  "Yangian"



2) Point modification of coherent sheaves on surfaces

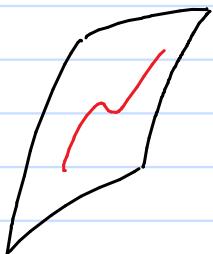
- \*  $H_*(\text{Coh}_S, \mathbb{C})$  for  $S/\mathbb{C} \rightsquigarrow$  "Affine Yangian of  $gl(1)$ "  
( $\rightsquigarrow$  Affine W-algebra of  $gl_r$ )



3) Curve modification of coherent sheaves on surfaces

- \*  $H_*(\text{Coh}_S, \mathbb{C})$  for  $S/\mathbb{C} \rightsquigarrow ?$

ex:  $\mathcal{C} = \mathbb{P}^1 \rightarrow$  "Affine Yangian  
 $\mathcal{C} \cdot \mathcal{C} = -2$  of  $gl(2)$ "



$\mathcal{C}$  = chain of  $\mathbb{P}^1$ 's  $\rightarrow$  "Affine Kac-Moody  
Yangian"

$\mathcal{C}$  = elliptic curve  $\rightarrow$ ? Pagoola Algebra?

## O - Motivation (Hecke operators)

•  $X$  : smooth projective curve /  $\mathbb{F}_q = k$

$$x \in X(k)$$

Def: A modification (of length  $l$ ) at  $x$  of a vector bundle

$\mathcal{N}$  on  $X$  is a subsheaf  $\mathcal{N}' \subset \mathcal{N}$  s.t. :

$$- \text{supp}(\mathcal{N}/\mathcal{N}') = \{x\}$$

$$- \text{length } (\mathcal{N}/\mathcal{N}') = l$$

Put  $H_{r,d} = \{F: \text{Bun}_{r,d}(\mathbb{F}_q) \rightarrow \mathbb{C} \mid |\text{supp}(F)| < \infty\}$

$$H_r = \bigoplus_{d \in \mathbb{Z}} H_{r,d}$$

Hecke operator:

$$T_{x,l} : H_{r,d} \rightarrow H_{r,d+l} \quad (l \geq 1)$$

$$T_{x,l}(F)(\mathcal{N}) = \sum_{\mathcal{N}'} F(\mathcal{N}') \quad ]$$

where  $\mathcal{N}'$  runs among all modifications of  $\mathcal{N}$  at  $x$  of length  $l$ .

Using a natural pairing on  $H_r$ , one defines dual Hecke operators  $T_{x,l}^*$ .

Classical (and important!) :  $\{T_{x,l}, T_{x,l}^* \mid l \in \mathbb{N}\}$

form a Family of commuting operators

("Hecke algebra at  $\infty$ ")

Better : . For fixed  $r$ ,

$$\text{End}(H_r) \cong \langle T_{x,l}, T_{x,l}^* \rangle \cong \mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]^{S_r}$$

$$(\cong \mathbb{C}[e_1, e_2, \dots, e_r^{\pm 1}])$$

. For all  $r$  at once

$$\prod_r \text{End}(H_r) \cong \langle T_{x,l}, T_{x,l}^* \rangle \cong \mathbb{C}[b_n \mid n \in \mathbb{Z}]$$

(Heisenberg algebra  
with zero central charge)

Rem : - One can vary  $\infty \mapsto$  crucial for (geometric)  
Langlands program

- Best formulation : Hall algebra

$$H_0 := \bigoplus_{d>0} \{f: \text{Coh}_{0,d}(\mathbb{F}_q) \rightarrow \mathbb{C}\}$$

$H_0$  is an algebra, via Hall product

$$f * g (\tau) = \sum_{\tau \leq \tau'} q^{\frac{1}{2} \langle \tau, \tau' \rangle} f(\tau/\tau') g(\tau')$$

↑ Euler form

$$\langle F, g \rangle = \text{hom}(F, g) - \text{ext}'(F, g)$$

$$\leadsto \cdot H_0 \cong \text{Heis}^+ = \mathbb{C}[b_n \mid n > 0]$$

•  $H_0$  acts on each  $H_r$ .

# I - COHA of points on curves (S-Vasserot)

Basic idea: replace  $\text{Fun}(\text{Bun}_r, \mathbb{C})$  by  $H^*(\text{Bun}_r, \mathbb{C})$

$$\hookrightarrow \underbrace{\text{Coh}_{0,l}}_{\substack{\text{smooth} \\ \text{stacks}}} \times \underbrace{\text{Bun}_{r,d}}_{\substack{\text{(as } \text{Ext}^2(-,-)=0)}} \xleftarrow{q} \widetilde{\text{Bun}}_{r,d;l} \xrightarrow{p} \widetilde{\text{Bun}}_{r,d+l}$$

where  $\cdot \text{Bun}_{r,d}, \text{Coh}_{0,l}, \dots$  now denote stacks

$\text{smooth stacks}$   $\cdot \widetilde{\text{Bun}}_{r,d;l}$  is the stack of length  $l$  modifications

$$\{ \mathcal{N}' \subset \mathcal{N} \mid \deg(\mathcal{N}/\mathcal{N}') = l, \text{rk}(\mathcal{N}') = \text{rk}(\mathcal{N}) \}$$

$$\cdot p(\mathcal{N}' \subset \mathcal{N}) = \mathcal{N} \quad (\text{proper})$$

$$q(\mathcal{N}' \subset \mathcal{N}) = (\mathcal{N}/\mathcal{N}', \mathcal{N}') \quad (\text{smooth}) \quad \begin{matrix} \text{Ext}^* \text{ v.b.} \\ \text{stack} \end{matrix}$$

$\rightsquigarrow$  Can define, for any  $c \in H^*(\text{Coh}_{0,d})$  a "Hecke" operator

$$T_c : H^*(\text{Bun}_{r,d}) \rightarrow H^*(\text{Bun}_{r,d+l})$$

Moreover, the algebra of such Hecke operators is

$$H_0 := \bigoplus_{d \geq 0} H^*(\text{Coh}_{0,d})$$

with Hall product induced by

$$\text{Coh}_{0,d} \times \text{Coh}_{0,d'} \xleftarrow{\quad} \widetilde{\text{Coh}}_{0,d+d'} \xrightarrow{\quad} \text{Coh}_{0,d+d'}$$

$\rightsquigarrow$  Questions: what is  $H_0$ ?

$$\text{Action on } H_r := \bigoplus_d H^*(\text{Bun}_{r,d}) ?$$

Recall:  $\Sigma \in \text{Coh}(\text{Coh}_{r,d} \times X)$  tautological coh. sheaf

$$\text{Set } c_{i,\gamma}(\Sigma) = \sum_{\gamma \in \Pi} c_{i,\gamma} \otimes \gamma^* \in H^*(\text{Coh}_{r,d}) \otimes H^*(X)$$

where  $\Pi = \{1, a_1, b_1, \dots, a_g, b_g, w\}$  is a basis  
of  $H^*(X)$

Theorem (Atiyah-Bott; Heinloth for  $\text{Coh}_{r,d}$ )

$$(i) H^*(\text{Coh}_{0,d}) \simeq S^d(H^*(\text{Coh}_{0,1})) \simeq S^d(\underbrace{H^*(X)[z]}_{\text{coh}_{0,1} \simeq X \times BG_m, \deg z = 2})$$

$$(ii) \forall r > 1, \quad H^*(\text{Coh}_{r,d}) = \mathbb{C}[c_{i,\gamma} \mid i > 1, \gamma \in \Pi]$$

Rem :  $c_{i,\gamma}(\Sigma) \mid_{Bun_{r,d}} = 0 \quad \text{for } i > r.$

Description of  $H_0$  : put  $\mathcal{J}\mathcal{L} = [\Delta_X] = \sum_{\gamma} \gamma \otimes \gamma^* \in H^*(X)^{\otimes 2}$

. As a shuffle algebra:

$$H_0 = \bigoplus_d S^d(H^*(X)[z]) \quad \text{ex: } (a_1 z)_1 + (a_1 z)_2 + \dots + (a_1 z)_d \in S^d$$

$$S^d \Rightarrow P(z) * Q(z) = \sum_{\substack{\text{shuffles} \\ \gamma \\ \gamma \\ \gamma}} \gamma \cdot \left\{ \sum_{\substack{1 \leq k \leq d \\ d+1 \leq l \leq d+e}} \left( 1 + \frac{\mathcal{J}\mathcal{L}}{z_l - z_k} \right) P(z_{[1,d]}) Q(z_{[d+1,d+e]}) \right\}$$

"weighted symmetrization op."

. By generators / relations:

Generators:  $\{\gamma z^l \mid \gamma \in \Pi, l > 1\}$

$$\text{Relations: } [\gamma z^l, \gamma' z^{l'}] = \mathcal{J}\mathcal{L} \cdot (\gamma \otimes \gamma') \frac{z_1^l z_2^{l'} - z_1^{l'} z_2^l}{z_1 - z_2}$$

analog of Drinfeld relation  
for Yangian (R-matrix)

$$1 + \frac{\mathcal{J}\mathcal{L}}{z}$$

Upshot:  $H_0$  is a symmetric, braided algebra, generated  
in degree 1 (by  $H^*(X)[z]$ ), and

$$[H_0 \underset{v.s}{\simeq} \text{Sym}(H^*(X)[z])]$$

## \* Action of $H_r$ on $H_r$

1) Identify  $H^*(\text{Coh}_{r,d})$  with  $\underbrace{S^\infty(H^*(x)[z])}_{\substack{\text{"H}^*(x)-\text{colored} \\ \text{symmetric functions"}}} = \varprojlim S^n(H^*(x)[z])$

via :  $\text{Coh}_{r,d-N} \times \text{Coh}_{0,N} \xrightarrow{\oplus} \text{Coh}_{r,d}$

$$\begin{aligned} H^*(\text{Coh}_{r,d}) &\xrightarrow{\oplus^*} H^*(\text{Coh}_{r,d-N}) \otimes H^*(\text{Coh}_{0,N}) \\ &\xrightarrow{\rho_r} H^*(\text{Coh}_{r,d-N}) \otimes H^*(\text{Coh}_{0,N}) \\ &\simeq S^n(H^*(x)[z]) \end{aligned} \quad \left. \right\}$$

Set  $p_{l,r} = \sum (\gamma z^l)_i = \gamma z^l \otimes 1 \otimes 1 \otimes \dots + 1 \otimes \gamma z^l \otimes 1 \otimes \dots + \dots$

$\rightsquigarrow H^*(\text{Coh}_{r,d}) \simeq \mathbb{C}[p_{l,r} \mid l \geq 0, \gamma \in \pi] =: H$

2) Consider the vertex operator

$$T(s) = \exp \left( \sum_{r,l} p_{l,r} \gamma_\infty^* s^{l+1} - \sum_l l p_{l-1,\omega} \omega_\infty s^{l+1} \right) \cdot \exp \left( - \sum_{r,l} \frac{\partial}{\partial p_{l,r}} \gamma_\infty s^{-l} \right) \in \text{End}(H) \otimes H^*(x)[[s^{\pm 1}]]$$

∞ component

Then, for any  $\begin{cases} P(z) \in H \simeq H^*(\text{Coh}_{r,d}) \\ \gamma z^l \in H^*(\text{Coh}_{0,1}) \end{cases}$

$$T_{\gamma z^l}(P(z)) = (T(s) \cdot P(z))_{[s^{l+1-r} \gamma_\infty^*]} \quad (*)$$

↑ coefficient of monomial

(this uses some Lemma of Negut)

$$\begin{cases} \pi = S^\infty \\ 1_d = [\text{Coh}_d] \\ \alpha \in \mathbb{Z}^2 \\ 1_{f,d} \times 1_{r,e} \end{cases}$$

Rem : ~ One can give a presentation of  $H := \bigoplus_{r \geq 0} H_r$  by gen. & relat°

- The (1d) COTHA  $H$  encodes many interesting enumerative geometry problems on  $\text{Coh}_{r,d}$  : relations in  $H^*(\text{Bun}_{r,d}^{\text{ss}})$ , Goller's weighted TQFT, intersection theory on Quot schemes, ...?

## II - COHA of points on a surface

(S-V, Minets, Y.Zhao, Kapranov-V.)

$S$ : smooth quasi-projective surface /  $\mathbb{C}$

$\text{coh}_{0,0,l} = \text{moduli stack of } 0\text{-dim sheaves on } S \text{ of length } l.$   
(singular)

$$\hookrightarrow \begin{array}{c} \text{"quasi-smooth"} \xrightarrow{q} \tilde{\text{coh}}_{*,l} \\ \text{coh}_{0,0,l} \times \text{coh}_{0,0,l'} \end{array} \xleftarrow{p} \text{coh}_{0,0,l+l'} \quad )$$

Thm (KV; Y.Zhao in K-theory, Minets for  $S = T^*X$ )

The space  $H_0 := \bigoplus_l H_*^{\text{B-M homology}}(\text{coh}_{0,0,l})$  has a natural Hall multiplication.

What is known (by me :))

(i) "PBW theorem":  $H_0 = H_0(S)$  carries a filtration s.t.

$$\left[ \text{gr}(H_0(S)) \cong \text{Sym} \left( H_*(S)[z_1, z_2] \right) \right] \quad (\text{KV})$$

[Compare with case of curves  $\text{gr}(H_0(X)) \cong \text{Sym}(H_*(X)[z])$ ]

(ii)  $\exists$  a morphism from  $H_0(S)$  to some shuffle algebra (Y.Zhao)

(Minets for  $T^*X$ )

This morphism is injective for  $S = T^*X$  (Minets)

(iii) Special case:  $S = \mathbb{A}^2$

•  $\mathbb{C}^* \times \mathbb{C}^*$ -equivariance turned on:

$$H_0 \cong Y_{z_1, z_2}^+(\widehat{\mathfrak{gl}_1}) \quad \text{affine Yangian of } \mathfrak{gl}(1) \quad (\text{SV})$$

$$\underset{\text{def of}}{\cong} U^+(\mathbb{C}[t, s^{\pm 1}])$$

(very) Heuristically:  $\begin{cases} t^n \mapsto c_1(\mathcal{E}_{0,0,1})^n \cap [\text{coh}_{0,0,1}(\mathbb{A}^2)] \\ s^e \mapsto [\text{coh}_{0,0,e}] \\ t^n \mapsto \text{multiplicat}^\circ \text{ by tautological classes} \end{cases}$

• no equivariance

$$H_0 \cong \text{Sym}(H_*(\mathbb{A}^2)[z_1, z_2]) \rightsquigarrow (\text{co})\text{commutative Hopf algebra} \quad (\text{KV})$$

Rem: - One can replace  $H_*$  by  $K_0, A_*, \dots$  (oriented Borel-Moore homology theories)

- Negut constructs an action of a (version of)  $\gamma(\hat{g}_1)$  on homology (rather, K-theory) of suitable moduli spaces of vector bundles on  $S$  [also features some vertex operators]

### III COHA of curves on surfaces

[Jt. work, in progress, with:  
Diaconescu, Sala, Vasserot]

- $S$  smooth quasi-projective surface
- $C \subset S$  reduced curve

- Ex:
- .  $P' \subset T^*P'$
  - $X \subset T^*X$        $X$  smooth proj. curve
  - .  $S = \widetilde{\mathbb{P}^2}/\rho$  resolution of Kleinian singularity  
 $C$  = exceptional divisor
  - .  $S$  elliptic surface  
 $C$  = exceptional fiber
  - .  $S = K3,$   
...
- } "local" curve CY setup

$\mathcal{M}_{e,\alpha}$  := moduli stack of coherent sheaves on  $S$ , of chern class  $\alpha$ , supported on  $C$

Rem: -  $\mathcal{M}_{e,\alpha} = \varinjlim_n \mathcal{M}_{e^{(n)}, \alpha}$  ↪ moduli stack of coh. sheaves on  $n^{\text{th}}$  neighbor of  $C$  in  $S$

$\Rightarrow \mathcal{M}_{e,\alpha}$  is an ind-stack

Moreover, each  $\mathcal{M}_{e^{(n)}, \alpha}$  is locally of finite type, but typically has  $\infty$ -many irreducible components

(simplest) ex: a chain of  $P'$ 's

...  ...

$$\hookrightarrow \mathcal{M}_{e,\alpha} \times \mathcal{M}_{e,\beta} \xleftarrow{\quad \tilde{\mathcal{M}}_{e,\alpha+\beta} \quad} \mathcal{M}_{e,\alpha+\beta}$$

Thm (KV; Sala-Porta)

$\exists$  an associative algebra structure on  $H_e := \bigoplus_{\alpha} H_*(\mathcal{M}_{e,\alpha})$

Rem: May fix a polarization  $\omega \in H^2(S, \mathbb{Z})$  and consider  $\mathcal{M}_{e,d}^{ss}$ .

Some general properties:

•  $S = T^*X : \mathbb{H}_e$  carries a filtration s.t.  $\text{gr.}(\mathbb{H}_e) \underset{\text{v.s.}}{\simeq} \text{Sym}(g_e^{\text{BPS}})$

for some Lie algebra  $g^{\text{BPS}} = \bigoplus_{\alpha} g_{\alpha}^{\text{BPS}}$  (Davison - Meinhardt)

Moreover  $\text{ch}(g^{\text{BPS}})$  is known (Kac polynomials of curves)

[expect the same for any 2CY  $S$ ?]

•  $S = T^*X : \mathbb{H}_e$  is generated by  $\langle [\mathcal{M}_{e,0,\alpha}] ; \alpha \rangle$  and the multiplication by taut. classes.

Question: how to compute  $\mathbb{H}_e$ ??

DS<sup>2</sup>V : (partial) answer in cases where  $\mathcal{C}$  is a union of (-2) rational curves intersecting transversally.

$$\textcircled{1} \quad S = T^* \mathbb{P}' \supset \mathbb{P}' = \mathcal{C} \quad \mathcal{M}_e \simeq \text{Higgs}_{\mathbb{P}'}^{\text{nilp}} \quad (\mathbb{C}^*)^2 \curvearrowright S$$

• semistable COMAs:

$$\textcircled{2} \quad \forall \ell \in \mathbb{Z}, \quad \mathbb{H}_e^{(\ell)} := \bigoplus_{[\alpha] = \ell} H_*^{(\alpha)}(\mathcal{M}_{e,d}^{ss})$$

We have, for  $\alpha = r([\mathbb{P}'] + l\delta)$ , class of point

$$\mathcal{M}_{e,d}^{ss} \simeq \{(\Sigma, \varphi) \mid \Sigma \simeq \Theta(\ell)^{\oplus r}, \varphi \in \text{Hom}(\Sigma, \Sigma(z)) = \{0\}\}$$

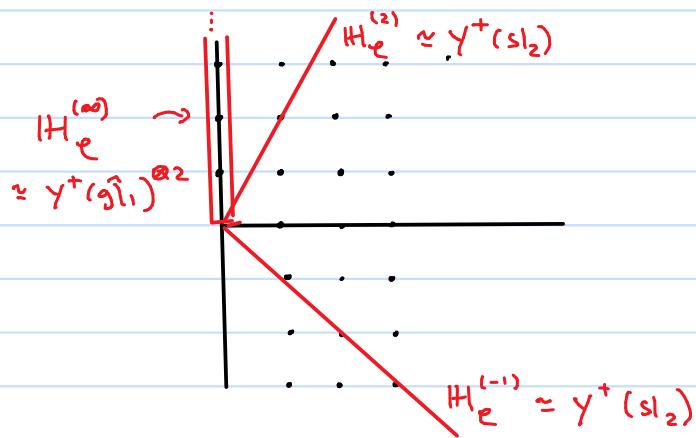
$$\simeq \text{BGL}_r$$

$$\Rightarrow \mathbb{H}_e^{(\ell)} \simeq \gamma_{\Sigma}^+ (S\Gamma_2) \underset{\text{v.s.}}{\simeq} \text{Sym}(S\Gamma_2^+ [n])$$

$$(b) \quad H_{\mathcal{E}}^{(\infty)} = \bigoplus_{k>0} H_k(M_{\mathcal{E}, 0, 0, k}) \simeq H_{\mathcal{E}, 0} \quad (\text{coHA of 0-dim sheaves})$$

$$\hookrightarrow H_{\mathcal{E}}^{(\infty)} \simeq Y_{\mathcal{E}}^+(\hat{\mathfrak{gl}}_1) \otimes H^*(\mathbb{P}^1)$$

Picture:



$$\hookrightarrow \text{Thm (DSV)} : H_{\mathcal{E}} \simeq Y^+(\hat{\mathfrak{gl}}_2)$$

[Warning: non-standard half "Drinfeld<sup>2</sup>" ]

$$\text{i.e. } g_{\mathbb{P}^1}^{\text{BPS}} \simeq \mathfrak{gl}_2[t, s^{\pm 1}]$$

$$(g_{\mathbb{P}^1}^{\text{BPS}})^+ \simeq \mathcal{H}[t, s^{\pm 1}] \oplus \mathcal{H}[t, s]$$

Idea of proof:

- use tilting equivalence & braid group actions to relate  $M_{\mathcal{E}}$  to representation stacks of quivers (preprojective algebras)
- relate braid group action on  $M_{\text{Rep } Q}$  to braid group actions on Yangians

$$\textcircled{2} \quad S = \widetilde{\mathbb{C}^2}/(\mathbb{Z}/3\mathbb{Z}) \quad \mathcal{E} = \text{exceptional divisor}$$



$$\text{Thm : (i) } H_{\mathcal{E}} \simeq Y_{\mathcal{E}}^+(\hat{\mathfrak{gl}}_3) \quad (\text{again, nonstandard half})$$

$$\mathcal{H}[t, s^{\pm 1}] \oplus \mathcal{H}[t, s]$$

$$\text{(ii) } H_{\mathcal{E}} \simeq Y_{\mathcal{E}}^+(\hat{\mathfrak{gl}}_2) \otimes Y_{\mathcal{E}}^+(\hat{\mathfrak{gl}}_2)$$

$$Y_{\mathcal{E}}^+(\hat{\mathfrak{gl}}_1)$$

③ Generalization:  $\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_r$

s.t.  $\begin{cases} \mathcal{C}_i \cdot \mathcal{C}_j = -2 \\ \mathcal{C}_i \cap \mathcal{C}_j = \emptyset \text{ or a point} \end{cases}$

Thm (Porta-Sala)  $H_{\mathcal{C}}$  only depends on formal neighborhood of  $\mathcal{C}$  in  $S$ .

Thm ("PBW")  $\bigotimes H_{\mathcal{C}_i} \xrightarrow{\sim} H_{\mathcal{C}}$

where the tensor product is taken over  $\gamma^+(\widehat{\mathfrak{gl}}_1)$  for each intersect point.

Cor:  $H_{\mathcal{C}}$  is isomorphic to the algebra generated by copies of  $\gamma^+(\widehat{\mathfrak{gl}}_2)$  (one for each  $\mathcal{C}_i$ ) , modulo **local** relations , one for each intersection .

Ex:  $S =$  elliptic surface with  $\mathcal{C} =$  cycle of  $\mathbb{P}^1$ 's



$\hookrightarrow H_{\mathcal{C}} \cong \gamma^+(\widehat{\mathfrak{gl}}_n)$  "triple loop algebra"

Next case :  $S = T^*\Sigma$   $\Sigma$  elliptic curve

this should give some version of  $\gamma^+(\widehat{\mathfrak{gl}}_1)$

("Pagoda algebra")

in progress