Kuznetsov components of Fano threefolds

Arend Bayer, University of Edinburgh

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Kuznetsov components of Fano threefolds

X = smooth Fano threefolds of Picard rank one: $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$
Index i_X: $K_X = -i_X \cdot H$.
Degree $d := H^3$.

Index $i_Y = 2$		Index $i_X = 1$	
d	Type	d	Type
4	$(2,2)$-c.i. $Y_4 \subset \mathbb{P}^5$	18	$X_{18} \subset \mathrm{Gr}(2,7)$
3	cubic threefold $Y_3 \subset \mathbb{P}^4$	14	Linear section $X_{14} = \mathrm{Gr}(2,6) \cap \mathbb{P}^9 \subset \mathbb{P}^{14}$
2	double quartic solid $Y_2 \twoheadrightarrow \mathbb{P}^3$	10	*Gushel-Mukai threefolds*

Definition A Gushel-Mukai threefolds is either

(ordinary) a $(2,1,1)$-complete intersection of $\mathrm{Gr}(2,5)$: $X_{10} = \mathrm{Gr}(2,5) \cap \mathbb{P}^7 \cap Q \subset \mathbb{P}^9$, or

(special) a double cover $X_{10} \twoheadrightarrow Y_5 = \mathrm{Gr}(2,5) \cap \mathbb{P}^6 \subset \mathbb{P}^9$ of a linear section of $\mathrm{Gr}(2,5)$, branched in a $(2,1,1,1)$-c.i., i.e. a GM K3 surface.

Kuznetsov components

$D(X) = D^b(X) = D^b(\mathrm{Coh}\, X) =$ bounded derived category of coherent sheaves.

Index $i_Y = 2$:

$$D(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(H) \rangle, \quad \text{i.e.,}$$

$$\mathcal{K}u(Y) = \{E \colon \mathrm{Hom}^\bullet(\mathcal{O}_Y, E) = \mathrm{Hom}^\bullet(\mathcal{O}_Y(H), E) = 0\}.$$

Index $i_X = 1$:

$$D(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle$$

where $\mathcal{E} =$ pull-back of tautological subbundle from $\mathrm{Gr}(2, ?)$.

Instead of an introduction: VBAC?

What does this have to do with vector bundles on algebraic curves??

Meta-Principle

Using $\mathcal{K}u(X)$ instead of $D^b(X)$ gives much more curve-like behaviour for Fano varieties of higher dimension.

Sometimes literally:
Theorem [BONDAL-ORLOV '95]
$Y_4 \subset \mathbb{P}^5$ (2,2)-c.i. $\rightsquigarrow$ pencil of quadrics $\rightsquigarrow$ 2:1-cover $C_2 \subset \mathbb{P}^1$. Then:

$$D^b(C_2) \cong \mathcal{K}u(Y_4).$$

Other times: Behaviour of $\mathcal{K}u(X)$ in terms of stability, Ext-vanishing, smoothness of moduli spaces, Brill-Noether behaviour, ...

Applications $X \rightsquigarrow \mathcal{K}u(X)$ highlights and encodes relations between different Fano threefolds.

More on Kuznetsov components

Kuznetsov components

Index $i_Y = 2$:

$$D^b(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(H) \rangle, \quad \text{i.e.,}$$

$$\mathcal{K}u(Y) = \{E \colon \mathrm{Hom}^\bullet(\mathcal{O}_Y, E) = \mathrm{Hom}^\bullet(\mathcal{O}_Y(H), E) = 0\}.$$

Index $i_X = 1$:

$$D^b(X) = \langle \mathcal{K}u(X), \mathcal{E}, \mathcal{O}_X \rangle$$

$\mathcal{E} =$ pull-back of taut. subbundle from $\mathrm{Gr}(2, ?)$.

Properties:

- $i \colon \mathcal{K}u(X) \subset D(X)$ is an *admissible* subcategory: i has left- and right adjoints $i^*, i^!$.
- These act quite naturally and geometrically.

Example:
$L \subset Y$ line $\rightsquigarrow i^* I_L = I_L$
$C \subset Y_3$ conic $\rightsquigarrow i^*\left(I_C(1)\right) = I_L^\vee(-1)$ where $L =$ line residual to C.

- $\mathcal{K}u(X)$ has *Serre functor* $\mathcal{S}_{\mathcal{K}u(X)}$:

$$\mathrm{Hom}(E, F) = \mathrm{Hom}(F, \mathcal{S}_{\mathcal{K}u(X)}(E))^\vee$$

- Ext-vanishing $\rightsquigarrow$ Objects in $\mathcal{K}u(X)$ have better-behaved deformation theory.
- $\mathcal{K}u(X)$ often controlled by Kuznetsov's *Homological Projective Duality*
- *Numerical K-group* of $\mathcal{K}u(X)$ has rank two.

Stability conditions on Kuznetsov components

Bridgeland stability = notion of stability that mimics slope-stability on curves as much as possible

$\mathcal{D} \subset D(X)$ admissible subcategory.
Definition A stability condition on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{A})$, where

- $Z\colon K(\mathcal{D}) \to \mathbb{C}$ group homomorphism,
- $\mathcal{A} \subset \mathcal{D}$ *heart of a bounded t-structure*, such that:

1. $Z(E) := Z(v(E)) \in \mathbb{H} \cup \mathbb{R}_{<0}$, for all nonzero $E \in \mathcal{A}$
2. Harder–Narasimhan filtrations exist in $\mathcal{A}$: for every $E \in \mathcal{A}$ there exists a (unique and functorial) filtration

$$E_0 \subset E_1 \subset \dots E_m = E$$

with E_i/E_{i-1} Z-semistable, $\mu(E_1/E_0) > \cdots > \mu(E_m/E_{m-1})$.

3. Support property

$\rightsquigarrow$ Slope: $\mu(E) = \frac{-\Re Z(E)}{\Im Z(E)}$
$\rightsquigarrow$ Definition: E is Z-semistable $\Leftrightarrow$ for all $A \subset E$, have $\mu(A) \leq \mu(E)$.

Theorem [B-Lahoz-Macri-Stellari '17, B-L-M-Nuer-Perry-S '19]

- Stability conditions exist on Kuznetsov components of all(*) prime Fano threefolds.
- They come with proper moduli spaces $M_\sigma(v)$ of semistable objects.
- This construction works in families of Fano threefolds.

Intermediate Jacobians

Definition

The intermediate Jacobian of a Fano threefold X is

$$J(X) := H^3(X, \mathbb{C})/(H^{2,1}(X, \mathbb{C}) \oplus H^3(X, \mathbb{Z}))$$

Intersection pairing on $H^3 \rightsquigarrow$ polarisation on $J(X) =$ polarised abelian variety.

$\mathrm{CH}^2_{\mathrm{hom}}(X)$ = 1-cycles $[C]$ of degree $H.[C] = 0$.
[BLOCH-SRINIVAS]: $\mathrm{CH}^2_{\mathrm{hom}}(X) = J(X)$.

Examples

- $J(Y_4) = J(C_2)$.
- [CLEMENS-GRIFFITHS] $J(Y_3)$ indecomposable and *not* Jacobian of a curve $\Rightarrow$ Y_3 not rational.

Fact Every moduli space $M_\sigma(v)$ of objects in $\mathcal{K}u(X)$ comes with a map

$$M_\sigma(v) \to J(X), \quad E \mapsto c_2(E) - c_2(E_0).$$

Theorem [PERRY] *Intrinsic* notion of intermediate Jacobians of categories (for admissible $\mathcal{D} \subset D(X)$):

$$J(\mathcal{K}u(X)) = J(X) \quad \text{for all Fano threefolds } X.$$

Torelli? Does $J(X)$ determine X?

- Does $J(X)$ determine $\mathcal{K}u(X)$?
- Does $\mathcal{K}u(X)$ determine X?

Examples and Applications

Key result

- (In all cases discussed today) Serre-invariant stability conditions on $\mathcal{K}u(X)$ are unique. [Feyzbakhsh-Pertusi, Jacovskis-Liu-Lin-Zhang]
- Applies to stability conditions constructed in [BLMS]. [Pertusi, Robinett, Yang]

Consequence, e.g. for Y_4, Y_3: Moduli spaces $M_\sigma(v)$ are smooth.

Examples (selected)

Y_3 cubic threefold:

- line $L \subset Y_3$. Then I_L stable, and $M_\sigma([I_L]) = F(Y_3)$ [Bernardara-Macri-Mehrothra-Stellari '09]
- $y \in Y_3 \Rightarrow i^* I_y$ is stable $\Rightarrow Y_3 \subsetneq M_\sigma(v)$. In fact: $M_\sigma(v) \to J(Y_3)$ is resolution of Θ-divisor: isomorphism outside $0 \in \Theta$, and $Y_3 \mapsto 0$. [B-Beentjes-Feyzbakhsh-Hein-Martinelli-Rezaee-Schmidt]

Corollary $\mathcal{K}u(Y_3)$ determines Y_3.
Also reproves classical Torelli: $J(Y_3)$ determines Y_3.

Y_2 double quartic solid, general. $M_\sigma(v) = Y_2 \cup \mathcal{C}$ reducible [Altavilla-Petkovic-Rota]
Corollary* $\mathcal{K}u(Y_2)$ determines Y_2.

Result Y_d Fano threefold of index two, $d \geq 2 \Rightarrow \mathcal{K}u(Y_d)$ determines Y_d.

Reconstructing X from $\mathcal{K}u(X)$

X prime Fano threefold of index 1, degree $d \geq 10$. Does $\mathcal{K}u(X)$ determine X? No!

Example $X_{10} \subset \mathrm{Gr}(2,5)$ Gushel-Mukai threefold
Classical period map: $\mathcal{M}_{10} \to \mathcal{P}, X_{10} \mapsto J(X_{10})$ [Debarre, Iliev, Manivel, Kuznetsov].
Fibers: at least two surfaces, "line transforms" and "conic transforms" of X_{10}, birational to X_{10}.

These period partners have the same $\mathcal{K}u(X_{10})$. [Kuznetsov-Perry]

Theorem [Jacosvkis-Liu-Zhang]

X_d prime Fano threefold of index 1, degree $d \geq 10$.

- $x \in X_d \Rightarrow i^*\mathcal{O}_x$ is stable, $\rightsquigarrow X_d \subsetneq M_\sigma(v)$.
- Brill-Noether reconstruction: $X_d = \{F \in M_\sigma(v) \colon \operatorname{ext}^2(F, i^!\mathcal{E}) = r\}$.
- $i^!\mathcal{E}$ is σ-stable.

By construction, have exact triangle $\mathcal{E}^{\oplus r} \to I_x \to i^*\mathcal{O}_x[-1]$ and hence $\operatorname{ext}^2(i^*\mathcal{O}_x, \mathcal{E}) = \operatorname{ext}^2(i^*\mathcal{O}_x, i^!\mathcal{E}) = r$.
Corollary X_d determined by $(\mathcal{K}u(X_d), i^!\mathcal{E})$.
Note Can interpret this via category $\langle \mathcal{K}u(X_d), \mathcal{E} \rangle$.
Special cases of this result (mostly in different language) observed previously [Mukai, Brambilla-Faenzi, Faenzi-Verra, Iliev-Markushevich-Tikhomirov]

Examples
X_{10}: $i^!\mathcal{E}$ can be in one of two 2-dimensional moduli space = surfaces identified by DIM.
X_{12}: $\mathcal{K}u(X_{12}) = D^b(C_7)$ and $i^!\mathcal{E} \mapsto \mathcal{O}_{C_7}$.
X_{12} obtained as Brill-Noether locus of stable rank two vector bundles on C_7 [Mukai]

Kuznetsov components via (enhanced) group actions

Conjecture [KUZNETSOV] $1 \leq d \leq 5$. There exists correspondene $\Gamma \subset \mathcal{M}_{Y_e} \times \mathcal{M}_{X_{4d+2}}$ with:

$$(Y_d, X_{4d+2}) \in \Gamma \Rightarrow \mathcal{K}u(Y_d) = \mathcal{K}u(X_{4d+2}).$$

Proved for $d \geq 3$. Yields explicit relations between Hilbert schemes on Y_d and X_{4d+2}.
[KUZNETSOV-PROKHOROV-SHRAMOV]

Theorem

In case $d = 2$:

- $\mathcal{K}u(Y_2) \neq \mathcal{K}u(X_{10})$ [SHIZHOU ZHANG, B-PERRY]
- But $\mathcal{K}u(Y_2)$ and $\mathcal{K}u(X_{10})$ are deformation-equivalent. [B-PERRY]

Key: both $\mathcal{K}u(Y_2)$ and $\mathcal{K}u(X_{10})$ are of Enriques type:
$S_{\mathcal{K}u_{Y_2}} = i \circ [2]$ where i is an involution.
How to take maximum advantage of this?

Group actions on categories

$\mathcal{D}$ linear category over a field, G finite group. What is an action of G on $\mathcal{D}$?

Level 1: $\operatorname{Aut}\mathcal{D}$ = autoequivalences of $\mathcal{D}$ up to natural isomorphism. Consider $G \to \operatorname{Aut}\mathcal{D}$
Issue: not enough to construct category of G-equivariant objects.

Level 2: [DELIGNE] Functor $\Phi_g \colon \mathcal{D} \to \mathcal{D}$ for all $g \in G$ and
natural transformation $\theta_{g,h} \colon \Phi_g \circ \Phi_h \to \Phi_{g\circ h}$ for all $g, h \in G$ satisfying associativity condition.
Not every action of Level 1 lifts to Level 2! [BECKMANN-OBERDIECK]

Definition $\mathcal{D}$ with action of G (Level 2).
$\mathcal{D}^G$ = invariant category = category of G-equivariant objects.
Example: G acting trivially on $\mathcal{D} = k$-Vect. Then $\mathcal{D}^G = G$-Rep.

Proposition (Residual action) [ELAGIN] G abelian. Then $\widehat{G}$ acts on $\mathcal{D}^G$ and $\left(\mathcal{D}^G\right)^{\widehat{G}} = \mathcal{D}$.

Examples [KUZNETSOV-PERRY] $Y_2 \twoheadrightarrow \mathbb{P}^3$ 2:1-cover branched at quartic K3 $S \subset \mathbb{P}^3$.
$\mathbb{Z}/2$ acts on $Y_2 \rightsquigarrow \mathbb{Z}/2$ acts on $D(Y_2) \rightsquigarrow \mathbb{Z}/2$ acts on $\mathcal{K}u(Y_2)$.
$\mathcal{K}u(Y_2)^{\mathbb{Z}/2} = D(S)$, with explicit formula for residual action on $D(S)$.

Special GM threefold: $X_{10} \twoheadrightarrow Y_5$ 2:1-cover branched at GM K3 $T \subset \operatorname{Gr}(2,5)$.
$\mathbb{Z}/2$ acts on $X_{10} \rightsquigarrow \mathbb{Z}/2$ acts on $D(X_{10}) \rightsquigarrow \mathbb{Z}/2$ acts on $\mathcal{K}u(X_{10})$.
$\mathcal{K}u(X_{10})^{\mathbb{Z}/2} = D(T)$, with explicit formula for residual action on $D(T)$.

Ordinary GM threefold: $\mathbb{Z}/2$-action exists on $\mathcal{K}u(X_{10})$.
$\mathcal{K}u(X_{10})^{\mathbb{Z}/2} = \mathcal{K}u(X_{10}^{\mathrm{op}})$ where X_{10}^{op}= associated (special) GM fourfold.

Enriques-K3 correspondence for categories

Recall: Q Enriques, with K3 cover $\widetilde{Q}$. Then:
$Q = \widetilde{Q}/(\mathbb{Z}/2)$, and so $D(\widetilde{Q})^{\mathbb{Z}/2} = D(Q)$.
$\mathbb{Z}/2$ acts on $D(Q)$ by tensoring with K_Q, and $D(Q)^{\mathbb{Z}/2} = D(\widetilde{Q})$.
This generalizes to categories!

Definition $\mathcal{D}$ is an Enriques category if it has a Serre functor with $S_{\mathcal{D}} = i \circ [2]$, where i = involution generating $\mathbb{Z}/2$-action.

Proposition

- If $\mathcal{D}$ is an Enriques category, then $\mathcal{D}^{\mathbb{Z}/2}$ is a CY-2 category (K3 cover).
- $\mathcal{D}_1, \mathcal{D}_2$ Enriques category. Then equivalence $\Phi\colon \mathcal{D}_1 \to \mathcal{D}_2$ lifts to a equivalence $\mathcal{D}_1^{\mathbb{Z}/2} \cong \mathcal{D}_2^{\mathbb{Z}/2}$, which is $\mathbb{Z}/2$-equivariant wrt residual action.

Theorem $\mathcal{K}u(Y_2) \neq \mathcal{K}u(X_{10})$, X_{10} special.
PROOF [B-PERRY]
Otherwise, get $\mathbb{Z}/2$-equivariant equivalence $D(S) \cong D(T)$, where D quartic K3, $T \subset \mathrm{Gr}(2,5)$ GM K3.
Derived Torelli for K3s $\Rightarrow$ get $\mathbb{Z}/2$-equivariant Hodge isometry $H^*(S,\mathbb{Z}) \cong H^*(T,\mathbb{Z})$
Lattice computation $\Rightarrow$ Picard group of T contains

$$\begin{pmatrix} 10 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 10 & 3 \\ 3 & 0 \end{pmatrix}.$$

Contradiction!

Deformation equivalence

Theorem $\mathcal{K}u(Y_2) \neq \mathcal{K}u(X_{10})$, X_{10} special.
PROOF Otherwise, ... Lattice computation $\Rightarrow$ Picard group of T contains

$$\begin{pmatrix} 10 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 10 & 3 \\ 3 & 0 \end{pmatrix}, \quad \text{contradiction!}$$

Theorem [B-PERRY] $\mathcal{K}u(Y_2) \sim \mathcal{K}u(X_{10})$

PROOF Idea: GM K3 T deforms to K3 with Picard group $\begin{pmatrix} 10 & 3 \\ 3 & 0 \end{pmatrix}$ = T_0 = quartic containing a line.

Consider family $\mathcal{T} \to C$ with special fibre = T_0, general fibre = GM K3.
Key: $\mathbb{Z}/2$-action extends to $D(\mathcal{T})$.
On T_0 conjugate to action coming from $Y_2 \twoheadrightarrow \mathbb{P}^3$ branched at T_0.
So $D(\mathcal{T})^{\mathbb{Z}/2}$ = family of categories over C with

- special fibre = $D(T_0)^{\mathbb{Z}/2} = \mathcal{K}u(Y_2)$, and
- other fiber = $D(T)^{\mathbb{Z}/2} = \mathcal{K}u(X_{10})$.

To make this rigorous, need group action on enhanced categories, e.g. stable ∞-categories.
Good news: Action at Level 2 normally lift to actions at Level ∞.

Corollary Moduli spaces in $\mathcal{K}u(Y_2)$ and $\mathcal{K}u(X_{10})$ are deformation-equivalent.