Commutativity in Several Contexts
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Character varieties
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Representation Varieties
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Topology and MHSs of Representation and Character Varieties of Abelian/Nilpotent groups

COMMUTATIVE SEQUENCES IN REDUCTIVE GROUPS

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Dedicated to Peter Newstead

Outline

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3. Representation Varieties for Reductive Groups
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Commuting matrices

- Fix $r, n \in \mathbb{N}$. Let

$$Comm_n^r := \{(A_1, \cdots, A_r) \in (Mat_{n \times n}\mathbb{C})^r \; : \; A_i A_j = A_j A_i\}$$

  Commuting variety of r-tuples of size n matrices.

- Commutativity is not so simple!! Many Open Problems:
  - Irreducibility?
  - Dimensions of components?
  - Geometry? Arithmetic?

- Known: (trivial cases: $Comm_1^r = \mathbb{C}^r$; $Comm_n^1 = Mat_{n \times n}\mathbb{C}$)
  - $Comm_n^2$ is irreducible of the expected dimension $n^2 + n$
  - $Comm_n^4$ is reducible, for every $n \geq 4$ (!)
  - $Comm_n^3$ is reducible for $n \geq 32$ (Guralnick, 1992; improved to $n \geq 29$)

The diagonal component

- **Distinguished component** in $Comm_n^r$: closure of those r-tuples that **can be simultaneously diagonalized.**

Let $D_n \subset Mat_{n \times n}\mathbb{C}$ be the vector space of diagonal matrices, and:

$$\phi : GL(n, \mathbb{C}) \times D_n^r \to Comm_n^r, \quad (g, B_1, \cdots, B_n) \mapsto (gB_1g^{-1}, \cdots, gB_ng^{-1})$$

Then

$$DCom_n^r := \overline{im(\phi)} \qquad \text{is irreducible}$$

Let's call it the "Diagonal Component" in $Comm_n^r$.

Theorem [Motzkin-Taussky, 1955] For $r = 2$

$$Comm_n^2 = DCom_n^2$$

so that $Comm_n^2$ is *irreducible* of dimension $n^2 + n$.

For larger r, $\exists$ more than one component in general.

Commuting r-tuples in groups

- Fix $r \in \mathbb{N}$, G a (compact, Lie) group. Let

  $$Comm^r(G) = \{(g_1, \cdots, g_r) \in G^r \; : \; g_i g_j = g_j g_i\}$$

  Space of (ordered) r-tuples (or r-sequences) of **commuting elements** in G.

V. Kac - A. Smilga, 1999: "The problem of constructing the quantum vacuum states of pure supersymmetric Yang-Mills theories placed on a small 3-dimensional spatial torus T^3 is reduced to a pure mathematical problem of classifying the flat connections on T^{3}"

If $r \geq 2$, $Comm^r(G)$ is not necessarily connected! Ex: $G = SO(3)$!! We have now a **torus component:** Let $T \subset G$ be a **fixed** maximal torus.

$$TCom^r(G) \quad := \quad \overline{im(\psi)} \subset Comm^r(G)$$

$$\psi : G \times T^r \to Comm^r(G) \qquad (g, t_1, \cdots, t_r) \mapsto (g t_1 g^{-1}, \cdots, g t_r g^{-1})$$

Theorem [Kac-Smilga '99] If $r > 2$, and G is simple, then $Comm^r(G)$ is connected (hence equals the torus component) only for $G = SU(n)$ or $G = Sp(n)$.

Commuting varieties - G reductive

- Finally, let G be a reductive (complex algebraic) group

$$Comm^r(G) := \{(g_1, \cdots, g_r) \; : \; g_i g_j = g_j g_i\} \subset G^r$$

  and call it Commuting variety of r-tuples in G.

It is an affine algebraic variety, since $G \subset GL(V)$ for some vector space V.

Example: write $GL_n \equiv GL(n, \mathbb{C})$.

$$Comm^r(GL_n) \;\; = \;\; Comm_n^r \cap (GL_n)^r$$
$$TCom^r(GL_n) \;\; = \;\; DCom_n^r \cap (GL_n)^r.$$

$Comm^r(G)$ is generally a very singular algebraic variety, again not necessarily irreducible (one component coming from max. torus), with intricate topology.

Remark: By contrast, the variety of commuting matrices $Comm_n^r$ is contractible!

Moduli Space of Commuting Matrices/Sequences in G

We wish to consider the space of *r-sequences of commuting linear operators on a vector space* $V \cong \mathbb{C}^n$ *without a preferred basis.*

- Fix $r, n \in \mathbb{N}$. $G = GL_n$ acts on $Comm_n^r$ by conjugation:

$$MC_n^r = Comm_n^r / G.$$

- Motivation: Representations of commutative algebras / Hilbert scheme of points on $\mathbb{C}^r$.

- Open problems: Irreducibility? Dimension of components? Geometry?

- Remarks: Again, there is a diagonal component for matrices: $DCom_n^r /\!\!/ G$, and a torus component in the group case $TCom^r(G) /\!\!/ G$.

- We need the (affine) **geometric invariant theory** (GIT) quotient: $MC_n^r := Comm_n^r /\!\!/ G$, and similarly:
$$MC^r(G) := Comm^r(G) /\!\!/ G \cong \hom(F_r, G) /\!\!/ G.$$

Then: stability corresponds to irreducible representations; proper stability to good reps, etc.

Character varieties

- Γ – a finitely presented group:

$$\Gamma = \langle \gamma_1, \cdots, \gamma_n \,|\, r_1, \cdots, r_m \rangle$$

  Ex: fundamental group $\pi_1(M)$ of a manifold/variety M.
- G – a Lie group.
  Typically, G is a real or complex reductive group
  Ex: $G = SL_n\mathbb{C},\ GL_n\mathbb{C},\ SL_n\mathbb{R},\ U(n),\ Sp_n,\ ...$
- $R_\Gamma G := \hom(\Gamma, G)$ – the G-representation variety of Γ
  (affine algebraic variety, given $G \subset GL_n\mathbb{C}$)
- $X_\Gamma G := \hom(\Gamma, G)/\!\!/ G$ – the G-character variety of Γ.

It is the GIT quotient, under *conjugation*: $g \in G$, $\rho \in \hom(\Gamma, G)$:

$$(g \cdot \rho)(\gamma) := g\,\rho(\gamma)\,g^{-1}, \qquad \gamma \in \Gamma.$$

Example: with $\Gamma = \mathbb{Z}^r$ (free abelian group) we have
$R_{\mathbb{Z}^r}\,G = Comm^r(G)$ and $X_{\mathbb{Z}^r}\,G = MC^r(G)$.

Motivation

- (Topology/Diff. Geometry) Space of Flat G-connections on a manifold M with $\pi_1(M) = \Gamma$.

- (Algebra) Matrix invariants under **simultaneous conjugation**.

- (Knot theory) The A-**polynomial** is defined by the image of a morphism between character varieties: $X_\Gamma \, SL_2\mathbb{C} \to X_{\mathbb{Z}^2} \, SL_2\mathbb{C}$.

- Non-abelian Hodge correspondence:

Theorem ([Hitchin, Donaldson, Corlette, Simpson 1986-90])

*Let M be a Riemann surface and G be real/complex reductive Lie group. Then, with $\Gamma = \pi_1(M)$, the character variety $X_\Gamma \, G = \hom(\Gamma, G)/\!\!/ G$ is **homeomorphic** to $\mathcal{H}_M \, G$, a moduli space of G-**Higgs bundles** over M.*

Summary of Topological and algebraic invariants

Let X be a space with finite cohomology (eg, a compact manifold, a finite CW complex, an algebraic variety, etc).

Let $b_k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$. The Poincaré polynomial of X is:

$$P_X(t) := \sum_{k \geq 0} b_k(X) \, t^k$$

Euler characteristic: $\chi(X) := P_X(-1) = \sum_{k \geq 0} (-1)^k \, b_k(X)$

Example: If G is a connected compact Lie group of positive dimension, then $\chi(G) = 0$.

Now, let X be a quasi-projective algebraic variety X. Its cohomology decomposes into "Hodge pieces" of dimensions $h^{k,p,q}(X)$, $k, p, q \in \{0, \cdots, 2d\}$. Mixed Hodge polynomial:

$$\mu_X(t, u, v) := \sum_{k,p,q} h^{k,p,q}(X) \, t^k u^p v^q.$$

Then: $P_X(t) = \mu_X(t, 1, 1)$ and the Serre (E^c-) polynomial is $E_X^c(u, v) := \mu_X^c(-1, u, v)$.

Examples and properties

Polynomial invariants associated to a **topological space**:

Space M	Poincaré polynomial $P_M(t)$	Euler char. $\chi(M)$
$\mathbb{R}^n$	1	1
Σ_g	$1 + 2gt + t^2$	$2 - 2g$
S^n	$1 + t^n$	$1 + (-1)^n$
$\mathbb{CP}^n$	$1 + t^2 + \cdots + t^{2n}$	$n + 1$
$X \times Y$	$P_X(t)\, P_Y(t)$	$\chi(X)\chi(Y)$
$X \sqcup Y$	?	$\chi(X) + \chi(Y)$

Polynomial invariants associated to a **quasi-projective variety**:

Space M	Mixed Hodge $\mu_X(t, u, v)$	Serre (E^c-) polynomial
Σ_g	$1 + gt(u + v) + t^2 uv$	$1 - gu - gv + uv$
$\mathbb{CP}^n$	$1 + t^2 uv + \cdots + t^{2n} u^n v^n$	$1 + uv + \cdots + (uv)^n$
toric	$\sum_{j=0}^{d} a_j (t^2 uv - 1)^j$	$\sum_{j=0}^{d} a_j (x - 1)^j; \quad x = uv$
$GL_n\mathbb{C}$	$\prod_{j=1}^{n}(1 + t^{2j-1} u^j v^j)$	$\prod_{j=1}^{n}(1 - x^{-j}) x^{n^2}$
$X \times Y$	$\mu_X\, \mu_Y$	$E_X^c(u, v)\, E_Y^c(u, v)$
$X \sqcup Y$	?	$E_X^c(u, v) + E_Y^c(u, v)$

Surface groups

Let $\Gamma =$ (central extension of) $\pi_1(\Sigma_g)$, the fund. group of a genus g compact orientable (Riemann) surface Σ_g.

- Hitchin ('87): Poincaré polynomials for $G = SL_2\mathbb{C}$
- Gothen ('94): Poincaré polynomials for $G = SL_3\mathbb{C}$

- Hausel - Rodriguez-Villegas ('08): Hodge-Deligne polynomials for $SL_2\mathbb{C}$, conjectures for higher n.
- Logares, Muñoz and Newstead ('13): E-polynomials for $SL_2\mathbb{C}$ and low g.
- Garcia-Prada, Heinloth and Schmitt ('14), Schiffman ('16), Mellit ('20): Poincaré polynomials for $G = GL_n\mathbb{C}$ and $G = SL_n$.
- …

Most of the above results are for smooth (twisted) character varieties.

Not much is known for $\hom(\pi_1\Sigma_g, G)/\!\!/ G$ except for low rank SL_n or GL_n, as these are very singular spaces.

Free groups: SDR and Topology

Let $\Gamma = F_r$ the free group of rank r, and note $X_{F_r} G \cong G^r /\!\!/ G$

Note that, by GIT, we can always embed
$X_{F_r} G = Specmax\ \mathbb{C}[G^r]^G$, as an affine subvariety of some $\mathbb{C}^N$. So we take usual Euclidean topology.

Theorem (F.-Lawton-Casimiro-Oliveira '09-'15)

Let G a real/complex reductive group, with maximal compact subgroup K. Then, $X_{F_r} K$ is a strong deformation retract of $X_{F_r} G$ (hence, Betti numbers agree $b_k(X_{F_r} K) = b_k(X_{F_r} G)$, for all k, r).

A concrete formula: Tom Baird, 2007, computed $P_{X_{F_r} SU(2)}(t)$:

$$P_{X_{F_r} SU(2)}(t) = 1 + t - \frac{t(1+t^3)^r}{1-t^4} + \frac{t^3}{2}\left(\frac{(1+t)^r}{1-t^2} - \frac{(1-t)^r}{1+t^3}\right)$$

As far as I know, in the very singular case of F_r, no computation was done for SL_n, $n > 2$.

Note: For F_r the representation variety $R_{F_r} G$ is trivially G^r !!

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Free groups: Cohomology and Serre polynomial

Again $\Gamma = F_r$ the free group of rank r, and consider $G = GL_n$, $DG = SL_n$ and $PG = PGL_n = GL_n/\mathbb{C}^*$.

Theorem (F-Nozad-Zamora '19)

We have isomorphisms: $H^*(X_{F_r} DG) \cong H^*(X_{F_r} PG)$ *as MHSs.*

Theorem (F-Nozad-Zamora '20)

The generating functions of Serre polynomials are related by a plethystic exponential:

$$1 + \sum_{n \geq 1} E^c_{X_{F_r} GL_n}(u, v) y^n = \text{PExp} \left[\sum_{n \geq 1} E^c_{X^{irr}_{F_r} GL_n}(u, v) y^n \right].$$

Open problems for general Γ, G: Irreducibility, Normality, Singularities, Topology, etc...

The case Γ abelian/nilpotent

From now on, Γ is **nilpotent**, that is, with $\Gamma_{k+1} = [\Gamma_k, \Gamma_k]$:

$$\Gamma = \Gamma_0 \rhd \Gamma_1 \rhd \cdots \rhd \Gamma_n = \{e\}.$$

Examples:

- $\Gamma = \mathbb{Z}^r$, free abelian; more generally, any abelian group.
- $\Gamma = H(\mathbb{Z})$, the (discrete) Heisenberg group; more generally, unipotent upper triangular matrices with $\mathbb{Z}$ entries.

Theorem (F.-Lawton '09-'14)

Let $\Gamma = \mathbb{Z}^r$ and G a complex reductive group, with maximal compact K. Then, $X_\Gamma K$ is a strong deformation retract of $X_\Gamma G$.

Theorem (Bergeron '15)

$X_\Gamma K$ is a strong deformation retract of $X_\Gamma G$, for any finitely generated nilpotent Γ.

Reduction from nilpotent to abelian

The abelianization of Γ is $\Gamma_{Ab} := \Gamma/[\Gamma, \Gamma]$, and we say that the *abelian rank* of Γ is $r \in \mathbb{N}_0$ when

$$\Gamma_{Ab} \cong \mathbb{Z}^r \oplus F, \qquad F \text{ finite abelian group.}$$

For K compact, let $R_\Gamma^0 K$ be the *torus component* of $R_\Gamma K = \hom(\Gamma, K)$.

Theorem (Bergeron-Silberman '16)

For Γ nilpotent of abelian rank r, and K compact Lie group:
$$R_\Gamma^0 K \cong R_{\mathbb{Z}^r}^0 K, \qquad \text{and} \qquad X_\Gamma^0 K \cong X_{\mathbb{Z}^r}^0 K.$$

This implies that $R_\Gamma^0 K$ and $X_\Gamma^0 K$ actually equal the torus component (Baird '09).

Theorem (F-Lawton-Silva '21)

For $\Gamma = \mathbb{Z}^r$, and G complex reductive, the torus component coincides with the identity component.

Char. var. of free Abelian groups - Irreducible components

Let G be reductive over $\mathbb{C}$, and T a fixed maximal torus.

Theorem (Sikora, F.-Lawton '15)

For $G = GL_n\mathbb{C}$, $SL_n\mathbb{C}$ and $Sp_n\mathbb{C}$,
$$X_{\mathbb{Z}^r}G \cong T^r/W.$$
G simple, and $X_{\mathbb{Z}^r}G$ irreducible $\Leftrightarrow$ $G = SL_n\mathbb{C}$ or $Sp_n\mathbb{C}$.

Corollary (F.-Lawton-Silva '21)

For $G = GL_n$, $G = SL_n\mathbb{C}$ or $G = Sp_n\mathbb{C}$ we have $X^0_{\mathbb{Z}^r}G = X_{\mathbb{Z}^r}G$ (= quotient of torus component).

Open question: Is $X_{\mathbb{Z}^r}G$ or $X^0_{\mathbb{Z}^r}G$ normal, for non-classical G?

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Char. var. of free Abelian groups - invariants

Let G be the complexification of a compact Lie group K.
Any $\sigma \in W$ acts on $\mathfrak{t}$, the Lie algebra of T.

Theorem (F-Silva, '18; F-Lawton-Silva '21)

Let X^0 be the torus component in $X_{\mathbb{Z}^r} G$. Then:
$$\mu_{X^0}(t, u, v) = \frac{1}{|W|} \sum_{\sigma \in W} \det(I + tuv M_\sigma)^r.$$

From the SDR, we recover:

Theorem (Stafa, '17)

Let $X^0_{\mathbb{Z}^r} K$ be the connected component of the identity in $X_{\mathbb{Z}^r} K = \hom(\mathbb{Z}^r, K)/K$ ("real" character variety). Then:
$$P_{X^0_{\mathbb{Z}^r} K}(t) = \frac{1}{|W|} \sum_{\sigma \in W} \det(I + t M_\sigma)^r.$$

$GL_n\mathbb{C}$ character varieties and symmetric products

Now take $G = GL_n\mathbb{C}$, the group of invertible matrices. It has $T_G = (\mathbb{C}^*)^n$ as **maximal torus**, and S_n as **Weyl group**:

$$X_{\mathbb{Z}^r}\,GL_n\mathbb{C} = MC^r(G) \cong T_G^r/W = ((\mathbb{C}^*)^n)^r/S_n = \mathrm{Sym}^n\left((\mathbb{C}^*)^r\right)$$

Similarly, for $K = U(n)$ we have $T_K = (S^1)^n$ and $W = S_n$ as well:

$$X_{\mathbb{Z}^r}\,U(n) = MC^r(K) \cong T_K^r/W = ((S^1)^n)^r/S_n = \mathrm{Sym}^n\left((S^1)^r\right)$$

Example

If $r = 2$ we have (over Σ_1 an elliptic curve):

$$X_{\mathbb{Z}^2}\,U(n) \;\cong\; \text{moduli space of rank } n \text{ vector bundles on } \Sigma_1$$
$$X_{\mathbb{Z}^2}\,GL_n \;\cong\; \text{moduli space of rank } n \text{ Higgs bundles on } \Sigma_1$$

Hence, these are homeomorphic, respectively, to an nth symmetric products of a real torus $(S^1)^2$, resp. of $(\mathbb{C}^*)^2$ (cf. Franco - Garcia-Prada - Newstead; Florentino - Biswas; ...).

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The generating function

For $G = GL_n$, since $X_{\mathbb{Z}^r} GL_n = X^0_{\mathbb{Z}^r} GL_n$ are symmetric products, we can use **Macdonald's / Cheah's formula.**

> **Theorem (Macdonald '62 / Cheah ' 94)**
>
> Let $P_X(t) = b_0 + b_1 t + b_2 t^2 + \cdots$. Then, $P_{Sym^n X}(t)$ is the coefficient of y^n in the rational function:
>
> $$\frac{(1 + ty)^{b_1}(1 + t^3 y)^{b_3} \cdots}{(1 - y)^{b_0}(1 - t^2 y)^{b_2} \cdots}$$

Cheah's formula generalizes this to the mixed Hodge polynomial. This means that the generating function $\sum_{n \geq 0} \mu_{X_{\mathbb{Z}^r} GL_n}(t, u, v)\, y^n$ is a plethystic exponential:

$$\sum_{n \geq 0} \mu_{X_{\mathbb{Z}^r} GL_n}(t, u, v)\, y^n = \prod_{k=0}^{d}(1 - (-tuv)^k)^{(-1)^k \binom{r}{k}} = \mathrm{PExp}((1 + tuv)^r y)$$

We have a recursion: Can get $\mu_{X_{\mathbb{Z}^r} GL_n}$ from all $\mu_{X_{\mathbb{Z}^r} GL_m}$, $m < n$.

The space of commuting r-tuples in K

Let K be a compact Lie group, and recall the space of commuting r-sequences: $\mathrm{R}_{\mathbb{Z}^r} K \equiv \hom(\mathbb{Z}^r, K)$ (not necessarily connected). By Kac-Smilga '99, $\mathrm{R}_{\mathbb{Z}^r} K$ is connected for $K = U(n)$ (disconnected in general for $K = SO(n)$ and other examples).

Theorem (Baird, '09)

If $K = SU(n)$, then:

$$P_{\mathrm{R}_{\mathbb{Z}^r} K}(t) = \begin{cases} \frac{1}{2}\left((1+t^2)(1+t)^r + (1-t^2)(1-t)^r\right), & n = 2 \\ \frac{1}{6}(1+2t^2+2t^4+t^6)(1+t)^{2r} + \frac{1}{2}(1-t^6)(1-t^2)^r \\ \quad + \frac{1}{3}(1-t^2-t^4+t^6)(1-t+t^2)^r, & n = 3 \end{cases}$$

Theorem: D. Ramras - M. Stafa formula ('17), with $d_1, \cdots, d_m$ the exponents of $Lie(K)$, $m = \mathrm{rk}K = \dim T_K$:

$$P_{\mathrm{R}_{\mathbb{Z}^r}^0 K}(t) = \frac{1}{|W|} \prod_{i=1}^{m} (1 - t^{2d_i}) \sum_{\sigma \in W} \frac{\det\left(I + t\, M_\sigma\right)^r}{\det\left(I - t^2\, M_\sigma\right)}.$$

Characteristic exponents

$$P_{R^0_{\mathbb{Z}^r},K}(t) = \frac{1}{|W|} \prod_{i=1}^{m} (1 - t^{2d_i}) \sum_{\sigma \in W} \frac{\det(I + t\, M_\sigma)^r}{\det(I - t^2\, M_\sigma)}.$$

Type	Group (compact)	$|W|$	exponents
A_n	$SU(n+1)$	$(n+1)!$	$2, 3, \cdots, n+1$
B_n	$SO(2n+1)$	$2^n n!$	$2, 4, \cdots, 2n$
C_n	$Sp(n)$	$2^n n!$	$2, 4, \cdots, 2n$
D_n	$SO(2n)$	$2^{n-1} n!$	$2, 4, \cdots, 2n-2, n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
E_8	E_8	696729600	$2, 8, 12, 14, 16, 20, 24, 30$

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The commuting variety of r-tuples in G

Now, denote by $R_\Gamma^0 G$ the torus component of $R_\Gamma G$ for any **nilpotent group** Γ of abelian rank r. We generalize Ramras-Stafa formula as follows (abbreviate $x = uv$).

Theorem (F.-Lawton, Silva '21)

For G reductive, all $r \geq 1$ we have:
$$\mu_{R_\Gamma^0 G}(t, x) = \frac{1}{|W|} \prod_{i=1}^{m} (1 - t^{2d_i} x^{d_i}) \sum_{g \in W} \frac{\det (I + tx A_g)^r}{\det (I - t^2 x A_g)}.$$

Additionally,
$$\frac{1}{|W|} \sum_{g \in W} \frac{\det (I + tx A_g)^r}{\det (I - t^2 x A_g)}$$

*is the mixed Hodge **series** for the G-equivariant cohomology:*
$H_G^*(R_\Gamma^0 G) \cong [H^*(T^r) \otimes H^*(BT)]^W$.

Corollary: The Euler characteristic of R_n^0 is zero for all $r, n > 0$.

A "trivial" example

Let $G = GL_n$, $r = 1$, and $\Gamma = \mathbb{Z}$. Then

$$R_\Gamma\, GL_n = \hom(\mathbb{Z}, GL_n) = GL_n$$

and the exponents are $d_i = i$ for $i = 1, 2, \cdots, n$. With $x = uv$:

$$\mu_{GL_n}(t, x) = \frac{1}{n!} \prod_{i=1}^{n}(1-t^{2i}x^i) \sum_{g \in S_n} \frac{\det\left(I + tx\, A_g\right)}{\det\left(I - t^2x\, A_g\right)} \overset{?}{=} \prod_{i=1}^{n}(1+t^{2i-1}x^i).$$

The $\overset{?}{=}$ follows from:
$$\frac{1}{n!} \sum_{\sigma \in S_n} \frac{\det\left(I - z\sigma\right)}{\det\left(I - q\,\sigma\right)} = \prod_{k=1}^{n} \frac{1 - zq^{k-1}}{1 - q^k},$$

a **generalization of the Molien formula** for the Hilbert-Poincaré series of the graded ring of invariant polynomials in n variables $\mathbb{C}[x_1, \cdots, x_n]^{S_n}$:

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A non-trivial (new) example:
2 commuting symplectic matrices

Let $G = Sp_2 = Sp(4, \mathbb{C})$, $\dim_{\mathbb{C}} G = 10$, rank 2, and exponents are $(2, 4)$. $W = D_4$, the diedral group of order 8 (symmetries of the square). Let $r = 2$, and $\Gamma_{Ab} = \mathbb{Z}^2$ (eg, 2 commuting $Sp(4, \mathbb{C})$ matrices):

$$R_\Gamma^0 G = Comm^2(Sp_2).$$

Mixed Hodge polynomial, with $x = uv$:

$$\mu_{R_\Gamma^0 G}(t, x) = 1 + t^2 x^2 + t^4 x^4 + 2(1 + t^4 x^4)(1 + t^2 x^2)t^3 x^2 + 2t^6 x^4 + 3t^{10} x^6.$$

Poincaré polynomial $(P(1) = 16)$:

$$P_{R_\Gamma^0 G}(t) = 1 + t^2 + t^4 + 2(t^3 + t^5 + t^6 + t^7 + t^9) + 3t^{10}.$$

E-polynomial $(\chi = 0)$

$$E_{R_\Gamma^0 G}(u, v) = 1 - x^2 - x^4 + x^6 = (1 - x^2)(1 - x^4).$$

The main diagram in the proof

G reductive with maximal compact K, maximal torus T, and Weyl group W. $T_K = T \cap K$. Γ nilpotent.

$$
\begin{array}{ccccc}
(G/T) \times_W T^r & \xrightarrow{\varphi_G} & \hom^0(\Gamma_{Ab}, G) & \longrightarrow & \hom^0(\Gamma, G) \equiv R^0_\Gamma G \\
\Big\uparrow & & FL \Big\uparrow & & PS \Big\uparrow \\
(K/T_K) \times_W T_K^r & \xrightarrow{\varphi_{K,\,B}} & \hom^0(\Gamma_{Ab}, K) & \xrightarrow{\cong BS} & \hom^0(\Gamma, K) \equiv R^0_\Gamma K.
\end{array}
$$

using the work of Pettet-Souto on the SDR
$\hom^0(\Gamma_{Ab}, K) \hookrightarrow \hom^0(\Gamma_{Ab}, G)$.

Some references

Thank you!
Happy Birthday Peter!

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