Higgs bundles and symmetric varieties

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VBAC Conference, University of Warwick, 25–29 July 2022
On the occasion of Peter Newstead's 80th birthday

Higgs bundles

- G semisimple complex Lie group
- $\theta : G \to G$ **holomorphic involution**
- $H := G^\theta$ fixed point subgroup (reductive)
- **Cartan decomposition** (eigenspace decomposition by θ)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$$

Isotropy representation

$$H \to \mathrm{GL}(\mathfrak{m})$$

- X compact Riemann surface of genus $g \geq 1$
- (G, θ)**-Higgs bundle on** X: Pair (E, φ)
  $E \to X$ H-bundle
  $\varphi \in H^0(X, E(\mathfrak{m}) \otimes K)$
  $E(\mathfrak{m}) = E \times_H \mathfrak{m}$ associated $\mathfrak{m}$-bundle via adjoint action
  K canonical line bundle of X

Higgs bundles

- There are stability criteria:
  The **moduli space of polystable (G,θ)-Higgs bundles** $\mathcal{M}(G,\theta)$ is the set of isomorphism classes of polystable (G,θ)-Higgs bundles
- The notion of (G,θ)-Higgs bundle generalizes the usual notion of G-Higgs bundle for a complex semisimple Lie group G (**Hitchin**):
  Consider the involution Θ of $G\times G$ given by

$$\Theta(g,h)=(h,g) \text{ for } g,h\in G$$

We have $(G\times G)^{\Theta}\cong G$ and $\mathfrak{m}\cong\mathfrak{g}$. Hence

$$(G\times G,\Theta)\text{-Higgs bundle} = G\text{-Higgs bundle}$$

We denote the moduli space of G-Higgs bundles by $\mathcal{M}(G)$

$$\mathcal{M}(G):=\mathcal{M}(G\times G,\Theta)$$

Real forms

- **Real form** of $G :=$ Fixed point subgroup $G^\sigma \subset G$ of an **antiholomorphic involution** $\sigma : G \to G$
- **Cartan**: Let (G, θ) be as above: There exists a **compact** antiholomorphic involution $\tau : G \to G$ (i.e. $G^\tau \subset G$ is a maximal compact subgroup) so that

$$\theta\tau = \tau\theta$$

- Define $\sigma := \theta\tau$. Then $G^\sigma \subset G$ is a real form of G
- All real forms arise in this way. More precisely:

$$\frac{\{\text{Holomorphic involutions}\}}{\text{conj. by (inner) automorph.}} \longleftrightarrow \frac{\{\text{Antiholomorphic involutions}\}}{\text{conj. by (inner) automorph.}}$$

Character varieties

- Let $G^\sigma \subset G$ be a real form
- **A representation of $\pi_1(X)$ in G^σ** is a homomorphism $\rho\colon \pi_1(X) \to G^\sigma$
- ρ is a **reductive representation** if composed with the adjoint representation in the Lie algebra of G^σ, decomposes as a sum of irreducible representations
  $\mathrm{Hom}^+(\pi_1(X), G^\sigma)$: set of reductive representations
- The **moduli space of representations** of $\pi_1(X)$ in G^σ or **G^σ-character variety** of X is defined to be the orbit space

$$\mathcal{R}(G^\sigma) = \mathrm{Hom}^+(\pi_1(X), G^\sigma)/G^\sigma$$

Non-abelian Hodge correspondence (Hitchin, Donaldson, Simpson, Corlette, ...)

Let θ and σ be as above. There is a homeomorphism

$$\mathcal{R}(G^\sigma) \cong \mathcal{M}(G, \theta)$$

We can regard G also as a real form of $G \times G$ corresponding to the antiholomorphic involution Σ of $G \times G$ given by

$$\Sigma(g, h) = (\tau(h), \tau(g)) \text{ for } g, h \in G,$$

where τ is a compact conjugation of G. Then

$$(G \times G)^\Sigma \cong G$$

And we recover the non-abelian Hodge correspondence for G:

$$\mathcal{R}(G) \cong \mathcal{M}(G)$$

Involutions in $\mathcal{M}(G)$ (Joint work with S. Ramanan)

- $\theta \in \mathrm{Aut}_2(G)$ defines an involution in $\mathcal{M}(G)$ given by

$$(E, \varphi) \mapsto (\theta(E), -\theta(\varphi)),$$

  where $\theta(E) := E \times_\theta G$
- This involution depends only on $a = \pi(\theta)$, where

$$\pi : \mathrm{Aut}(G) \to \mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$$

  Hence, we will denote it by

$$\iota_a : \mathcal{M}(G) \to \mathcal{M}(G)$$

- By extension of structure group one has a (finite) map

$$\mathcal{M}(G, \theta) \to \mathcal{M}(G)$$

  This can be proved by applying the Hitchin–Kobayashi correspondence relating the polystability to the existence of solutions to **Hitchin's equations**. Finiteness is derived from the fact that the quotient of the normalizer of G^θ in G by G^θ is a finite group

Involutions in $\mathcal{M}(G)$

- Let $\widetilde{\mathcal{M}}(G,\theta) \subset \mathcal{M}(G)$ be the image of this map. One has

$$\widetilde{\mathcal{M}}(G,\theta) \subset \mathcal{M}(G)^{\iota_a}$$

- For the converse, define for $\theta, \theta' \in \mathrm{Aut}(G)$ the relation

$$\theta' \sim \theta \Longleftrightarrow \theta' = \mathrm{Int}_g\, \theta\, \mathrm{Int}_{g^{-1}} \quad \text{for some} \quad g \in G$$

- The map $\pi : \mathrm{Aut}(G) \to \mathrm{Out}(G)$ descends to a map

$$\mathrm{Aut}(G)/\sim \to \mathrm{Out}(G)$$

- In particular, this defines the **clique map**

$$cl : \mathrm{Aut}_2(G)/\sim \to \mathrm{Out}_2(G)$$

Involutions in $\mathcal{M}(G)$

- **Theorem** (G-Ramanan). Let $\mathcal{M}_*(G) \subset \mathcal{M}(G)$ be the subvariety of smooth points. Then

$$\mathcal{M}_*(G)^{\iota_a} \subset \bigcup_{[\theta] \in cl^{-1}(a)} \widetilde{\mathcal{M}}(G, \theta)$$

- To determine $cl^{-1}(a)$, for $\theta \in \mathrm{Aut}_2(G)$ with $\pi(\theta) = a$, consider the subvariety

$$S_\theta = \{s \in G \mid s\theta(s) \in Z(G)\}$$

- G acts on S_θ by a **θ-twisted action** $g * s := gs\theta(g)^{-1}$
-

$$cl^{-1}(a) = S_\theta/(G \times Z(G)) \cong H^1_\theta(\mathbb{Z}/2, G/Z(G))$$

In particular, $cl^{-1}(a)$ is finite.

Involutions in $\mathcal{M}(G)$

- The subvarieties $\widetilde{\mathcal{M}}(G,\theta) \subset \mathcal{M}(G)$ are **Lagrangian** with respect the natural holomorphic symplectic structure of $\mathcal{M}(G)$ determined by the holomorphic structure of X. In the terminology used in the Langlands duality and mirror symmetry of Higgs bundle moduli spaces, they are the suport of (B,A,A)-**branes**.
- By considering the involution of $\mathcal{M}(G)$ given by $(E,\varphi) \mapsto (\theta(E),\theta(\varphi))$ as fixed points we obtain also the support of (B,B,B)-branes.
- Today we won't consider this except for the following beautiful "support" of a (B,B,B)-brane...

Involutions in $\mathcal{M}(G)$

Multiplicative (G, θ)-Higgs bundles

Joint work with **Guillermo Gallego**. Some parts of the project involve also **Ngô Bao Châu**

- G semisimple complex Lie group
- $\theta : G \to G$ holomorphic involution
- $H := G^\theta$ fixed-point subgroup
- $M := G/H$ **symmetric variety**
- X compact Riemann surface of genus $g \geq 0$
- $\{x_1, \cdots, x_n\}$ finite set of X and $X' = X \setminus \{x_1, \cdots, x_n\}$
- A **multiplicative (G, θ)-Higgs bundle** on X with **singular points** $\{x_1, \cdots, x_n\}$ is a pair (E, φ)
  E H-bundle over X
  $\varphi \in H^0(X', E(M)|_{X'})$
  where $E(M) = E \times_H M$ associated M-**fibre bundle** with H acting on $M = G/H$ by left multiplication

Multiplicative (G, θ)-Higgs bundles

- If we consider $(G \times G, \Theta)$ as above, recall that $(G \times G)^{\Theta} \cong G$ and then

$$M = \frac{G \times G}{(G \times G)^{\Theta}} \cong G$$

- In this situation, E is a G-bundle, $E(M)$ is the **adjoint bundle of groups**, and φ is an **automorphism** of $E|_{X'}$.
- We refer to multiplicative $(G \times G, \Theta)$-Higgs bundles simply as **multiplicative G-Higgs bundles**
- These objects have been studied by
  - **Hurtubise–Markman**, 2002 (Sklyanin integrable systems)
  - **Frenkel–Ngô**, 2011; **Bouthier**, 2015; **Jingren Chi**, 2017 (Geometrization of the trace formulas)
  - **Charbonneau–Hurtubise**, 2011; **Mochizuki** and **Yoshino**, 2017 (Singular monopoles on $X \times S^1$)
  - **Elliot–Pestun**, 2019 (Supersymmetric gauge theories)

Meromorphic data at the singularities

Need to introduce several ingredients:

- Consider for formal variable z, the following $\mathbb{C}$-algebras

  $\mathcal{O} = \mathbb{C}[[z]]$ **ring of formal power series**
  $\mathcal{K} = \mathbb{C}((z))$ **field of formal Laurent series**
  $\mathbb{D} = \mathrm{Spec}(\mathcal{O})$ **formal disk**
  $\mathbb{D}^* = \mathrm{Spec}(\mathcal{K})$ **formal puntured disk**

- Given any $\mathbb{C}$-scheme S, consider

  $S(\mathcal{O}) = \mathrm{Hom}_{\mathbb{C}-\mathrm{sch}}(\mathbb{D}, S)$ **positive formal loop space**
  $S(\mathcal{K}) = \mathrm{Hom}_{\mathbb{C}-\mathrm{sch}}(\mathbb{D}^*, S)$ **formal loop space**

- For $S = G$

  $G(\mathcal{O})$ **positive formal loop group**
  $G(\mathcal{K})$ **formal loop group**
  $\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ **affine Grassmanian** of G
  Gr_G is equipped with a natural **left** $G(\mathcal{O})$**-action**

Meromorphic data at the singularities

- $B \subset G$ **Borel subgroup**
- $T \subset B$ **maximal torus**
- $\Lambda_T = \mathrm{Hom}(\mathbb{C}^*, G)$ **coweights** associated to T
- $\Lambda_T^+ \subset \Lambda_T$ **dominant coweights** associated to B and T

As a consequence of the **"Cartan decomposition"** given by **Iwahori–Matsumoto** one has the following:

Theorem

The $G(\mathcal{O})$-orbits of Gr_G are parametrized by the elements of Λ_T^+

This decomposition states that any $g \in G(\mathcal{K})$ can be written as

$$g = g_1 z^\lambda g_2$$

for $g_1, g_2 \in G(\mathcal{O})$, $\lambda : \mathbb{C}^* \to T$ a coweight and z^λ the image of $z \in \mathbb{C}((z))$ under the map

$$\lambda(\mathcal{K}) : \mathbb{C}^*(\mathcal{K}) = \mathcal{K}^* \to T(\mathcal{K}) \subset G(\mathcal{K})$$

Meromorphic data at the singularities

- Let (E, φ) be a multiplicative G-Higgs bundle on X with singular points $\{x_1, \cdots, x_n\}$
- By restricting φ to a formal neighbourhood of a singular point x_i we obtain an element of $G(\mathcal{K})$. This element is well-defined up to the action of $G(\mathcal{O})$ on the left and right and so it determines a well defined double coset μ_i in

$$G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}) = G(\mathcal{O})\backslash \mathrm{Gr}_G$$

There are appropriate stability notions, and fixing $\mu = (\mu_1, \cdots, \mu_n)$ with $\mu_i \in \Lambda_T^+$ we can define the **moduli space**

$$\mathscr{M}(G, \mu)$$

of multiplicative G-Higgs bundles with singular points $\{x_1, \cdots, x_n\}$ and **meromorphic data** given by $\mu = (\mu_1, \cdots, \mu_n)$

Meromorphic data at the singularities

What about the meromorphic data for a multiplicative (G, θ)-Higgs bundle? For this we need few more ingredients:

- Consider (G, θ) as above for $\theta \in \mathrm{Aut}_2(G)$
- Let $H = G^\theta$ and $M = G/H$ be the symmetric variety
- Take the maximal torus T to be θ-invariant and let $A \subset T$ be a maximal θ-**split** torus (i.e. $\theta(a) = a^{-1}$ for $a \in A$)
- $\Lambda_A^+ \subset \Lambda_A = \mathrm{Hom}(\mathbb{C}^*, A)$ **dominant coweights** of A
- Consider the **loop space** $M(\mathcal{K})$ and the action of $G(\mathcal{O})$ on the left

Theorem (Nadler)

There is a bijection between $G(\mathcal{O})\backslash M(\mathcal{K})$ and the set of dominant coweights Λ_A^+

Meromorphic data at the singularities

- Let now (E, φ) be a multiplicative (G, θ)-Higgs bundle.
- Again, by by restricting φ to a formal neighbourhood of a singular point x_i we obtain an element of $M(\mathcal{K})$ which is only well-defined up to the action of $H(\mathcal{O})$ on the left.That is we get an element $H(\mathcal{O})\varphi_{x_i}$ in $H(\mathcal{O})\backslash M(\mathcal{K})$
- But $H(\mathcal{O})\varphi_{x_i}$ is contained in the orbit $G(\mathcal{O})\varphi_{x_i}$ and hence defines an element in $G(\mathcal{O})\backslash M(\mathcal{K})$, which by the Nadler bijection defines an element $\lambda_i \in \Lambda_A^+$
- Fixing the meromorphic data $\lambda = (\lambda_1, \cdots, \lambda_n)$ with $\lambda_i \in \Lambda_A^+$, we can define the **moduli space**

$$\mathscr{M}(G, \theta, \lambda)$$

of multiplicative (G, θ)-Higgs bundles with singular points $\{x_1, \cdots, x_n\}$ and **meromorphic data** given by $\lambda = (\lambda_1, \cdots, \lambda_n)$

Multiplicative Higgs bundles and involutions

- Since $A \subset T$ we have a natural map

$$m : \Lambda_A^+ \to \Lambda_T^+$$

- Let $\lambda_i \in \Lambda_A^+$ and $\mu_i := m(\lambda_i)$
- The map

$$(E, \varphi) \mapsto (\theta(E), \theta(\varphi)^{-1})$$

defines an involution $\iota_a : \mathscr{M}(G, \mu) \to \mathscr{M}(G, \mu)$, where $a = \pi(\theta) \in \mathrm{Out}_2(G)$

- There is a map

$$\mathscr{M}(G, \theta, \lambda) \to \mathscr{M}(G, \mu)$$

and, as you may suspect, the image is in the fixed point locus $\mathscr{M}(G, \mu)^{\iota_a}$

- To make sense of this, we need some **background on symmetric varieties**

Multiplicative Higgs bundles and involutions

- Recall $S_\theta = \{s \in G \mid s\theta(s) \in Z(G)\}$, and the θ-twisted action of G on S_θ given by $g * s := gs\theta(g)^{-1}$
- The following subset $S^\theta \subset S_\theta$ is G-invariant

$$S^\theta = \{s \in G \mid \theta(s) = s^{-1}\}$$

Theorem (Richardson)

1. The set of θ-twisted orbits of S^θ is finite $(= H^1_\theta(\mathbb{Z}/2, G))$
2. The orbit of the neutral element $M^\theta := G * 1$ is a closed affine subvariety of G
3. The map

$$\begin{aligned} G &\longrightarrow M^\theta \\ g &\longmapsto g\theta(g)^{-1} \end{aligned}$$

induces an isomorphism of affine G-varieties $G/G^\theta \to M^\theta$.

Multiplicative Higgs bundles and involutions

- We can now make sense of extension of structure group to obtain a map

$$\mathscr{M}(G,\theta,\lambda) \to \mathscr{M}(G,\mu)$$

where $\mu = m(\lambda)$ for $m : \Lambda_A^+ \to \Lambda_T^+$

Theorem (Gallego–G)

- Let $\widetilde{\mathscr{M}}(G,\theta,\lambda)$ be the image of this map. Then

$$\widetilde{\mathscr{M}}(G,\theta,\lambda) \subset \mathscr{M}(G,\mu)^{\iota_a}$$

- Let $\mathscr{M}_*(G,\mu) \subset \mathscr{M}(G,\mu)$ be the subvariety of smooth points. Then

$$\mathscr{M}_*(G,\mu)^{\iota_a} \subset \bigcup_{[\theta]\in cl^{-1}(a), m(\lambda)=\mu} \widetilde{\mathscr{M}}(G,\theta,\lambda)$$

Meromorphic data and Langlands duality

- Let $\check{G}$ be the **Langlands dual group** of G. This group can be realized as the dual group of a certain **Tannkian category** over the affine Grassmanian Gr_G.
- Given $T \subset B \subset G$ as above, there are similar subgroups $\check{T} \subset \check{B} \subset \check{G}$ so that

$$\Lambda_T = \mathrm{Hom}(\mathbb{C}^*, T) \cong \mathrm{Hom}(\check{T}, \mathbb{C}^*)$$

(dom.) coweights for $G \longleftrightarrow$ **(dom.) weights** for $\check{G}$

- Using also Tannakian theory, **Nadler** associated to (G, θ) (or the corresponding symmetric space G/H), a dual group $\hat{H}$. This group, known as the **Nadler group** is a reductive subgroup of the Langlands dual $\check{G}$.
- **dominant coweights** $\Lambda_A^+ \longleftrightarrow$ **dominant weights** for $\hat{H}$

Hitchin fibration

- Let $T \subset G$ as above and $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of T. Let W be the corresponding **Weyl group**. One has the **Chevaley morphism** $\mathfrak{g} \to \mathfrak{t}/W$ and the **Chevaley restriction** theorem

$$\mathfrak{g} /\!\!/ G \cong \mathfrak{t}/W$$

- Coming back to the familiar G-Higgs bundles over X, by means of the Chevaley morphism one can define the **Hitchin map** and the **Hitchin integrable system**

$$\mathcal{M}(G) \to B(G)$$

  where $B(G) = H^0(X, \mathfrak{t} \otimes K/W)$ is the **Hitchin base**
- **Spectral curve** description of fibres (**Hitchin**)
- **Cameral curve** description of fibres and **gerbe structure** (**Donagi**, **Donagi–Gaitsgory**, **Ngô** and others...)

Hitchin fibration

- Consider now (G, θ), and recall $H = G^\theta$ and the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$
- Let $\mathfrak{a} \subset \mathfrak{m}$ a **maximal abelian subalgebra** (Lie algebra of A)
- $W(\mathfrak{a}) = N_H(\mathfrak{a})/C_H(\mathfrak{a})$ **small Weyl group**
- Now one has the **Kostant–Rallis** morphism $\mathfrak{m} \to \mathfrak{a}/W(\mathfrak{a})$ and the corresponding **Kostant–Rallis** restriction theorem

$$\mathfrak{a} /\!\!/ H \cong \mathfrak{a}/W(\mathfrak{a})$$

- From which one can define the Hitchin map

$$\mathcal{M}(G, \theta) \to B(G, \theta)$$

where $B(G, \theta) = H^0(X, \mathfrak{a} \otimes K/W(\mathfrak{a}))$

Hitchin fibration

- This map has been studied by **Peón**, **G–Peón–Ramanan**, **Schaposnick, Hitchin–Schaposnick**, ...
- Spectral curve description of the fibres for some classical groups have been given by **Schaposnick, Hitchin–Schaposnick** and others...
- Cameral curve description and gerbe structure for **quasi-split groups** has been given by **G–Peón** (**abelianization**)
- The description for non quasi-split groups is much more involved and the fibres are non-abelian (**Hitchin–Schaposnick**, **Peón** and **Franco–G–Newstead** — for elliptic curves)
- What about the **multiplicative** Hitchin map?

Hitchin fibration

Part of the main ingredients are the following:

- For multiplicative G-Higgs bundles one uses the **multiplicative version of the Chevaley** restriction theorem given by
$$G /\!\!/ G \cong T/W$$
where $T \subset G$ is a maximal torus
- For multiplicative (G, θ)-Higgs bundles we need a **multiplicative version of the Kostant–Rallis** restriction theorem. This is precisely provided by

Theorem (Richardson)

Consider (G, θ) and, as above, let $H = G^\theta$ and let $M^\theta \subset G$ be, the θ-twisted G-orbit of the unit element of G. Let $A \subset T$ be a maximal θ-split subtorus. Then

$$M^\theta /\!\!/ H \cong A/W(A)$$

Homework for Peter

- The Hitchin map for multiplicative G-Higgs bundles has been first studied by **Hurtubise–Markman** when X is an **elliptic curve**. In this situation $\mathscr{M}(G,\mu)$ is a **completely integrable Hamiltonian system**
- **Homework**: Give an explicit description of $\mathscr{M}(G,\mu)$ when X is an elliptic curve, along the lines of the explicit description of $\mathcal{M}(G)$ given in this case by **Franco–G-Newstead**

THANK YOU!

and

HAPPY BIRTHDAY PETER!