Orthogonal bundles of low rank and quadric fibrations

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Motivation: Enumeration of maximal subbundles

Let C be a complex projective smooth curve of genus $g \geq 2$. Let V be a general stable bundle of rank r and degree d over C.

For certain values of m and g, there are a finite number of rank m subbundles of maximal degree in V. When it is finite, we denote this number $N(g, r, d, m)$.

- Segre, Nagata, Lange, Ghione: $N(g, 2, d, 1) = 2^g$
- Oxbury, Okonek–Teleman, Teixidor: $N(g, r, d, 1) = r^g$
- Teixidor, Lange–Newstead: $N(g, r, d, 2)$
- Holla (2004): $N(g, r, d, m)$ in full generality

Symplectic and orthogonal bundles

Let $L \to C$ be a line bundle. A bundle $V \to C$ is *L-valued symplectic* (resp., *L-valued orthogonal*) if there is a bilinear nondegenerate skew-symmetric (resp., quadratic) form $\sigma \colon V \otimes V \to L$.

A subbundle $E \subset V$ is *isotropic* if $\sigma(E \otimes E) = 0$.

For certain choices of m and g, a general symplectic or orthogonal V has a finite number of isotropic subbundles of rank m and maximal degree.

D. Cheong, I. Choe & GH (2021) computed this number for rank n isotropic subbundles of symplectic bundles of rank $2n$.

Orthogonal case: Work in progress. However, we can try to solve the problem in low rank using structure theorems for orthogonal bundles.

Orthogonal bundles of rank up to four

In "Theta characteristics of an algebraic curve" (1971), Mumford gives a list of $\mathcal{O}_C$-valued orthogonal bundles for rank ≤ 4.

1. Line bundle M of order two: $M \otimes M \xrightarrow{\sim} \mathcal{O}_C$
2. $M \oplus M^{-1}$ for any line bundle M
3. Pushdown of line bundles in the Prym of an étale double cover of C
4. $\mathrm{Sym}^2 E \otimes \det(E^*)$ for E of rank two
5. $E_1 \otimes E_2$ with $\det(E_1) \cong \det(E_2^*)$, with form given by

$$(E_1 \otimes E_2) \otimes (E_1 \otimes E_2) \; \rightarrow \; \wedge^2 E_1 \otimes \wedge^2 E_2.$$

Can we use these to answer the enumeration questions in low rank?

Orthogonal bundles and quadric fibrations

Definition

For V orthogonal of rank r, we define

$$Q_V \ := \ \mathbb{P}\{v \in V : \sigma(v \otimes v) = 0\} \ \subset \ \mathbb{P}V,$$

a bundle of smooth quadric hypersurfaces of dimension $r - 2$ over C.

Lemma

A subbundle $E \subseteq V$ is isotropic if and only if $\mathbb{P}E$ is contained in Q_V.

Thus the study of isotropic subbundles of V is equivalent to the study of projective bundles contained in the quadric bundle Q_V.

The rank four case

Suppose $V = E_1 \otimes E_2$ for rank two bundles E_1, E_2 with $\det(E_1) \cong \det(E_2)^*$. The quadratic form is given by the projection

$$(E_1 \otimes E_2) \otimes (E_1 \otimes E_2) \ \to \ \wedge^2 E_1 \otimes \wedge^2 E_2 \ \cong \ \mathcal{O}_C.$$

For such a V, we have $Q_V = \mathbb{P}E_1 \times_C \mathbb{P}E_2$, the decomposable locus.

Isotropic subbundles of orthogonal bundles of rank four

A subbundle whose projectivisation is contained in $\mathbb{P}E_1 \times_C \mathbb{P}E_2$ must be

$$N_1 \otimes N_2, \qquad N_1 \otimes E_2, \quad \text{or} \quad E_1 \otimes N_2,$$

where $N_i \subset E_i$ is a line subbundle.

Such a subbundle has maximal degree among all such subbundles if and only if each N_i is maximal in E_i. As $N(g, 2, d, 1) = 2^g$, we get:

Proposition

Suppose that E_1 and E_2 are general stable bundles with $\deg(E_i) \equiv g - 1 \mod 2$. Then $E_1 \otimes E_2$ has

- $2 \cdot 2^g$ rank two isotropic subbundles of maximal degree $-(g-1)$, and
- 4^g isotropic line subbundles of maximal degree $-(g-1)$.

The second Stiefel–Whitney class

Let V be an $\mathcal{O}_C$-valued orthogonal bundle of rank r and trivial determinant. Then V is the associated vector bundle of some principal SO_r-bundle, unique up to orientation.

Definition

The *second Stiefel–Whitney class* $w_2(V) \in H^2(C, \mathbb{Z}/2) = \mathbb{Z}/2$ is the obstruction to lifting an SO_r-structure on V to a Spin_r-structure.

Two characterisations of $w_2(V)$

Serre (1990)

For any theta characteristic κ on C, we have

$$w_2(V) \equiv h^0(C, V \otimes \kappa) + r \cdot h^0(C, \kappa) \mod 2.$$

Choe–H. (2014, 2015)

Suppose V has rank $2n$ or rank $2n+1$. If $E \subset V$ is an isotropic subbundle of rank n, then $\deg(E) \equiv w_2(V) \mod 2$.

Moduli spaces of orthogonal bundles

- $\mathcal{MO}^0_C(r;\mathcal{O}_C) \subset \mathcal{SU}_C(r,\mathcal{O}_C)$: semistable $\mathcal{O}_C$-valued orthogonal bundles of rank r and trivial determinant, with $w_2 = 0$

- $\mathcal{MO}^1_C(r;\mathcal{O}_C) \subset \mathcal{SU}_C(r,\mathcal{O}_C)$: semistable $\mathcal{O}_C$-valued orthogonal bundles of rank r and trivial determinant with $w_2 \neq 0$; that is, *not* arising from Spin_r-bundles

- $\mathcal{MO}_C(2n;\mathcal{O}_C(x)) \subset \mathcal{SU}_C(2n,\mathcal{O}_C(nx))$: semistable $\mathcal{O}_C(x)$-valued orthogonal vector bundles of rank $2n$ and determinant $\mathcal{O}_C(nx)$, for any $x \in C$

(If $V \cong V^* \otimes \mathcal{O}_C(x)$, then $\det(V)^{\otimes 2} \cong \mathcal{O}_C(rx)$, so r must be even.)

Each of these is projective and irreducible, of dimension $\frac{1}{2}r(r-1)(g-1)$.

Maps of moduli spaces

Theorem

There are surjective finite maps of moduli spaces

$$\Theta_4^0 \colon \mathcal{SU}_C(2, \mathcal{O}_C) \times \mathcal{SU}_C(2, \mathcal{O}_C) \ \to \ \mathcal{MO}_C^0(4; \mathcal{O}_C),$$

$$\Theta_4^1 \colon \mathcal{SU}_C(2, \mathcal{O}_C(x)) \times \mathcal{SU}_C(2, \mathcal{O}_C(-x)) \ \to \ \mathcal{MO}_C^1(4; \mathcal{O}_C)$$

for any $x \in C$, and

$$\Phi_4 \colon \mathcal{SU}_C(2, \mathcal{O}_C) \times \mathcal{SU}_C(2, \mathcal{O}_C(x)) \ \to \ \mathcal{MO}_C(4; \mathcal{O}_C(x))$$

given by $(E_1, E_2) \mapsto E_1 \otimes E_2$.

Maximal rank two isotropic subbundles of a general element:

	$\mathcal{MO}_C^0(4, \mathcal{O}_C)$	$\mathcal{MO}_C^1(4, \mathcal{O}_C)$	$\mathcal{MO}_C(4, \mathcal{O}_C(x))$
g even	∞ of degree $-g$	$2 \cdot 2^g$ of degree $1 - g$	2^g of degree $2 - g$
g odd	$2 \cdot 2^g$ of degree $1 - g$	∞ of degree $-g$	2^g of degree $2 - g$

Isotropic Quot schemes

If V is any vector bundle, we denote the Quot scheme of subbundles of rank m and degree e by $\mathrm{Quot}^{\circ}_{m,e}(V)$.

Definition

If V is L-valued orthogonal, we denote by $\mathrm{IQ}^{\circ}_{m,e}(V)$ the closed sublocus of $\mathrm{Quot}^{\circ}_{m,e}(V)$ parametrising isotropic subbundles.

We will only consider only saturated subsheaves for now.

In general, there are many pathologies (Cheong–Choe–H. 2021, Int. J. of Math.)

Orthogonal bundles of rank three

Following Mumford (1971), we have:

Theorem

(a) There is a surjective finite map
$$\Theta_3^0 \colon \mathcal{SU}_C(2, \mathcal{O}_C) \ \to \ \mathcal{MO}_C^0(3; \mathcal{O}_C)$$
given by $E \mapsto \mathrm{Sym}^2 E$.

(b) For any $x \in C$, there is a surjective finite map
$$\Theta_4^1 \colon \mathcal{SU}_C(2, \mathcal{O}_C(-x)) \ \to \ \mathcal{MO}_C^1(3; \mathcal{O}_C).$$
given by $E_1 \mapsto \mathrm{Sym}^2 E_1 \otimes \mathcal{O}_C(x)$.

Recall that $\mathcal{MO}_C(3; \mathcal{O}_C(x))$ is empty.

Isotropic Quot schemes in rank three

For simplicity, suppose $w_2(V) = 0$, so $V = \mathrm{Sym}^2 E$ for E of trivial determinant.

It is easy to see that Q_V is the relative Segre embedding $\mathbb{P}E \hookrightarrow \mathbb{P}\mathrm{Sym}^2 E$.

Isotropic line subbundles are of the form $N^{\otimes 2}$ where $N \subset E$ is a line subbundle.

Proposition

The map $N \mapsto N^{\otimes 2}$ defines an isomorphism

$$\mathrm{Quot}^\circ_{1,d}(E) \xrightarrow{\sim} \mathrm{IQ}^\circ_{1,2d}(\mathrm{Sym}^2 E).$$

In particular, if g is odd and E is general in $\mathcal{SU}_C(2, \mathcal{O}_C)$, then V has 2^g isotropic line subbundles of maximal degree $-(g-1)$.

Notice that all isotropic line subbundles of V have even degree.

The structures we have encountered in rank three and four are (loosely speaking) consequences of the exceptional isomorphisms

$$\mathsf{Spin}_4 \xrightarrow{\sim} \mathsf{SL}_2 \times \mathsf{SL}_2 \quad \text{and} \quad \mathsf{Spin}_3 \xrightarrow{\sim} \mathsf{SL}_2.$$

But there are also exceptional isomorphisms

$$\mathsf{Spin}_6 \xrightarrow{\sim} \mathsf{SL}_4 \quad \text{and} \quad \mathsf{Spin}_5 \xrightarrow{\sim} \mathsf{Sp}_4.$$

What can we say about the structure of orthogonal bundles in ranks five and six?

Isotropic Quot schemes?

Enumeration of maximal isotropic subbundles?

Orthogonal bundles of rank six

Observation: Suppose W is a bundle of rank four and determinant L. Then

$$(\wedge^2 W) \otimes (\wedge^2 W) \; \to \; \wedge^4 W \; = \; L$$

makes $\wedge^2 W$ into a rank six L-valued orthogonal bundle.

Theorem (Structure of rank six $\mathcal{O}_C$-valued orthogonal bundles)

(a) The operation $W \mapsto \wedge^2 W$ defines a finite surjective map

$$\Theta_6^0 \colon \mathcal{SU}_C(4, \mathcal{O}_C) \; \to \; \mathcal{MO}_C^0(6, \mathcal{O}_C).$$

(b) Fix $x \in C$. Then $W \mapsto \mathcal{O}_C(x) \otimes \wedge^2 W$ defines a finite surjective map

$$\Theta_6^1 \colon \mathcal{SU}_C(4, \mathcal{O}_C(-2x)) \; \to \; \mathcal{MO}_C^1(6, \mathcal{O}_C).$$

Theorem (Structure of rank six $\mathcal{O}_C(x)$-valued orthogonal bundles)

Fix $x \in C$. Then $W \mapsto \wedge^2 W$ defines a finite surjective map

$$\Phi_6 \colon \mathcal{SU}_C(4, \mathcal{O}_C(x)) \to \mathcal{MO}_C(6, \mathcal{O}_C(x))$$

How can we "invert" these maps? And what if V is not stable?

Observation: If $F \subset W$ is of rank three, then $\wedge^2 F \subset \wedge^2 W$ is a rank three isotropic subbundle.

Given any rank six L-valued orthogonal V with a rank three isotropic subbundle, we will show how to construct W so that V is a twist of $\wedge^2 W$.

A structure theorem

Let V be L-valued orthogonal of rank six. Suppose $E \subset V$ is a rank three isotropic subbundle, so V is an extension

$$0 \ \to \ E \ \to \ V \ \to E^* \otimes L \ \to \ 0.$$

Moreover (H. 2007), the extension class $\delta(V)$ can be assumed to belong to

$$H^1(C, L^{-1} \otimes \wedge^2 E) \ \subset \ H^1(C, L^{-1} \otimes E \otimes E).$$

Thus $\delta(V)$ also defines a rank four extension

$$0 \ \to \ \wedge^2 E \ \to \ W \ \to \ L \ \to \ 0.$$

Proposition

We have $V \cong \wedge^2 W \otimes \det(E)^{-1}$.

Remark

Suppose that $L = \mathcal{O}_C$, that V is stable and that E has even degree. If $M \in \sqrt{\det(E)}$, then $W \otimes M^{-1}$ is a point of $\left(\Phi_6^0\right)^{-1}(V)$.

Quadric fibrations in rank six

Suppose W is of rank four and determinant L.

We have the relative Plücker embedding of $\mathrm{Gr}(2, W)$ in $\mathbb{P}\left(\wedge^2 W\right)$ as the projectivisation of the decomposable locus

$$\{\omega \in \wedge^2 W : \omega \wedge \omega = 0\}.$$

But this is exactly the condition that ω be isotropic with respect to the orthogonal structure on the rank six bundle $\wedge^2 W$.

Therefore, $Q_{\wedge^2 W} = \mathrm{Gr}(2, W)$, a fibration in smooth quadrics of dimension four.

Rank three isotropic subbundles of rank six orthogonal bundles

Suppose for simplicity that $V = \wedge^2 W$ for some W of rank four and trivial determinant.

Suppose that $\mathbb{P}E$ is a $\mathbb{P}^2$-subbundle of $\mathbb{P}(\wedge^2 W)$ contained in $\mathrm{Gr}(2, W)$.

By [Griffiths–Harris, Chap. 6], for each $p \in C$, either

(i) there exists $F_p = \mathbb{C}^3 \subset W|_p$ such that

$$\mathbb{P}E|_p \ = \ \{\Lambda \in \mathrm{Gr}\,(2, W|_p) : \Lambda \subset F_p\} \ = \ \mathrm{Gr}(2, F_p),$$

or

(ii) there exists $N_p = \mathbb{C} \subset W|_p$ such that

$$\mathbb{P}E|_p \ = \ \{\Lambda \in \mathrm{Gr}\,(2, W|_p) : N_p \subset \Lambda\}.$$

By local triviality, then, either

(i) there exists a uniquely defined rank three subbundle $F \subset W$ such that $E = \wedge^2 F$, or

(ii) there exists a uniquely defined line subbundle $N \subset W$ such that

$$E \; = \; \{v \wedge w : v \in N \text{ and } w \in W\}.$$

In case (i), $\deg(E) = 2 \cdot \deg(F)$ is even.

In case (ii), we have a presentation

$$0 \to N \otimes N \; \to \; N \otimes W \; \xrightarrow{\tau_N} \; E \; \to \; 0$$

where $\tau_N(v \otimes w) = v \wedge w$. Thus $\deg(E) = 2 \cdot \deg(N)$ is even.

Theorem (Isotropic Quot schemes for $L = \mathcal{O}_C$ and $w_2 = 0$)

Let W be a bundle of rank four and trivial determinant. For each d, there is an isomorphism

$$\mathrm{Quot}^{\circ}_{3,d}(W) \sqcup \mathrm{Quot}^{\circ}_{1,d}(W) \xrightarrow{\sim} \mathrm{IQ}^{\circ}_{3,2d}(\wedge^2 W),$$

given by

$$\begin{cases} E \mapsto \wedge^2 E & E \subset W \text{ of rank three and degree } d \\ N \mapsto \tau_N(N \otimes W) & N \subset W \text{ of rank one and degree } d. \end{cases}$$

In particular, $\mathrm{IQ}^{\circ}_{3,e}(\wedge^2 W)$ is disconnected for infinitely many e.

A similar statement applies for $w_2(V) \neq 0$. In this case the isotropic subbundles all have odd degree.

In the same way, one can show:

Theorem (Isotropic Quot schemes for $L = \mathcal{O}_C(x)$)

Let W be a bundle of rank four and determinant $\mathcal{O}_C(x)$. Then $E \mapsto \wedge^2 E$ defines an isomorphism

$$\mathrm{Quot}^{\circ}_{3,d}(W) \xrightarrow{\sim} \mathrm{IQ}^{\circ}_{3,2d}(\wedge^2 W),$$

and $N \mapsto \tau_N(N \otimes W)$ defines an isomorphism

$$\mathrm{Quot}^{\circ}_{1,d-1}(W) \xrightarrow{\sim} \mathrm{IQ}^{\circ}_{3,2d-1}.$$

Theorem (Popa–Roth 2003)

For $d \ll 0$, the Quot scheme $\mathrm{Quot}^{\circ}_{m,d}(W)$ is irreducible.

Corollary

For W as above, $\mathrm{IQ}^{\circ}_{3,e}(\wedge^2 W)$ is irreducible for $e \ll 0$.

Enumeration of maximal isotropic subbundles of rank three

Recall that $N(g, 4, d, 1) = 4^g$.

Let V be a rank six orthogonal bundle which is general in its component of moduli.

$\underline{L = \mathcal{O}_C}$: Suppose that either

- $g \equiv 1 \mod 4$ and $w_2(V) = 0$; or
- $g \equiv 3 \mod 4$ and $w_2(V) = 1$.

Then V has $2 \cdot 4^g$ rank three isotropic subbundles of maximal degree $-\frac{3}{2}(g-1)$.

$\underline{L = \mathcal{O}_C(x)}$: Suppose that $g \equiv 2 \mod 4$. Then V has 4^g rank three isotropic subbundles of maximal degree $3 - \frac{3g}{2}$.

Enumeration of maximal isotropic subbundles of rank two

Using similar arguments, one can prove:

Proposition

Suppose $g \equiv 1 \mod 12$, and suppose W is general in $\mathcal{SU}_C(4, \mathcal{O}_C)$. Then $\wedge^2 W$ has 12^g rank two isotropic subbundles of maximal degree $-\frac{5}{3}(g-1)$.

Note that these are never maximal as vector subbundles, as any bundle of rank six and degree zero has a rank two subbundle of degree at least $-\frac{4}{3}(g-1)$.

Enumeration of maximal isotropic line subbundles

A isotropic line subbundle of $\wedge^2 W$ corresponds to a section of $\mathrm{Gr}(2, W)$; equivalently a rank two subbundle of W.

Maximal isotropic line subbundles of $\wedge^2 W$ correspond to maximal rank two subbundles of W. The latter have been enumerated by Teixidor and by Holla.

Orthogonal bundles of rank five with $w_2 = 0$

Let W be a rank four bundle with a symplectic form $\alpha \colon \wedge^2 W \to \mathcal{O}_C$.

Then $\det(W)$ is trivial, and $\wedge^2 W$ is $\mathcal{O}_C$-valued orthogonal of rank six.

Now $S_W := \mathrm{Ker}(\alpha)$ is of rank five and determinant $\mathcal{O}_C$.

One can show that the restriction of the orthogonal structure on $\wedge^2 W$ to S_W is nondegenerate, and that $\wedge^2 W$ is an orthogonal direct sum $S_W \oplus \mathcal{O}_C$.

Structure of rank five orthogonal bundles

We write $\mathcal{MS}_C(2n; M)$ for the moduli space of semistable M-valued symplectic bundles over C. It is a closed irreducible subvariety of $\mathcal{SU}_C(2n, M^n)$.

Theorem

(a) The association $W \mapsto S_W$ defines a finite surjective map

$$\Theta_5^0 \colon \mathcal{MS}_C(4; \mathcal{O}_C) \ \to \ \mathcal{MO}_C^0(5; \mathcal{O}_C).$$

(b) Fix $x \in C$. The association $W \mapsto S_W \otimes \mathcal{O}_C(x)$ defines a finite surjective map

$$\Theta_5^1 \colon \mathcal{MS}_C(4; \mathcal{O}_C(-2x)) \ \to \ \mathcal{MO}_C^1(5; \mathcal{O}_C).$$

An inverse construction

Suppose V is $\mathcal{O}_C$-valued orthogonal of rank five and trivial determinant, and $w_2(V) = 0$ (not necessarily semistable).

Then $V \oplus \mathcal{O}_C$ is $\mathcal{O}_C$-valued orthogonal of rank six and trivial determinant with $w_2(V \oplus \mathcal{O}_C) = 0$.

By the rank six case, $V \oplus \mathcal{O}_C = \wedge^2 W$ for some W of rank four and trivial determinant.

Since the summand $\mathcal{O}_C$ is nowhere isotropic, $\mathbb{P}\mathcal{O}_C \cap \mathrm{Gr}(2, W)$ is empty.

Thus the image of $\mathcal{O}_C \to \wedge^2 W$ is everywhere indecomposable, and defines an antisymmetric isomorphism $W^* \xrightarrow{\sim} W$. Thus W is a symplectic bundle. Moreover,

$$V \oplus \mathcal{O}_C \;\cong\; \wedge^2 W \;\cong\; S_W \oplus \mathcal{O}_C.$$

By uniqueness of direct sum decomposition, $V \cong S_W$.

Quadric fibrations and rank five orthogonal bundles

As the orthogonal structure on V is inherited from that on $V \oplus \mathcal{O}_C = \wedge^2 W$, we have

$$Q_V \ = \ \mathbb{P}V \cap \mathrm{Gr}(2, W).$$

Since $V = \mathrm{Ker}(\alpha \colon \wedge^2 W \to \mathcal{O}_C)$, this is the Lagrangian Grassmann bundle of α-isotropic two-dimensional subspaces of fibres of W.

Isotropic Quot schemes in rank five

Suppose $V = S_W$ for a rank four $\mathcal{O}_C$-valued symplectic bundle W. Let $E \subset V$ be a rank two isotropic subbundle.

By [Griffiths–Harris, chapter 6], for each $p \in C$, there exists a line $N_p \subset W|_p$ such that

$$\mathbb{P}E|_p \ = \ \{\Lambda \in \mathrm{LG}(2, W|_p) : N_p \subset \Lambda \subset N_p^\perp\}.$$

By local triviality, there exists a uniquely defined line subbundle $N \subset W$ such that

$$E \ = \ \{v \wedge w : v \in N, w \in N^\perp\}.$$

We have a presentation $0 \to N \otimes N \to N \otimes N^\perp \xrightarrow{\lambda_N} E \to 0$, where $\lambda_N(v \otimes w) = v \wedge w$.

In particular, $\deg(E) = 2 \cdot \deg(N)$ is even.

Isotropic Quot schemes in rank five

Proposition

The map $N \mapsto \mathrm{Im}(\lambda_N)$ defines an isomorphism

$$\mathrm{Quot}^{\circ}_{1,d}(W) \xrightarrow{\sim} \mathrm{IQ}^{\circ}_{2,2d}(S_W).$$

Conjecture

Suppose $g \equiv 1 \mod 4$ and suppose $W \in \mathcal{MS}_C(4; \mathcal{O}_C)$ is a general point. Then S_W has 4^g rank two isotropic subbundles of maximal degree $-\frac{3}{2}(g-1)$.

We know that $N(g, 4, d, 1) = 4^g$, but we don't know if symplectic bundles are general in this sense.

Isotropic line subbundles in rank five

We have natural bijections

$$\{\text{isotropic line subbundles of } S_W\} \leftrightarrow \{\text{sections of } Q_{S_W} = \mathrm{LG}(W)\} \leftrightarrow \{\text{rank two } \alpha\text{-isotropic subbundles of the symplectic bundle } W\}$$

Let $\mathrm{LQ}^\circ_d(W)$ be the *Lagrangian* Quot scheme of rank two α-isotropic subbundles of degree d in W.

Proposition

For each d, the association $F \mapsto \wedge^2 F$ defines an isomorphism

$$\mathrm{LQ}^\circ_d(W) \xrightarrow{\sim} \mathrm{IQ}^\circ_{1,d}(S_W).$$

Enumerating maximal isotropic line subbundles

Theorem (Cheong–Choe–H. (2021))

Suppose W is general in $\mathcal{MS}_C(4; \mathcal{O}_C)$.

(a) If g is even then W has $2^{g-1}(3^g - 1)$ maximal rank two α-isotropic subbundles of degree $-(g-1)$.

(b) If g is odd then W has $2^{g-1}(3^g + 1)$ maximal rank two α-isotropic subbundles of degree $-(g-1)$.

By the previous slide, this number coincides with the number of maximal isotropic line subbundles of S_W.

Where to from here?

Main enumeration question

Let V be an orthogonal bundle which is general in its component of moduli. Compute the number of maximal isotropic subbundles of rank m in V, when this is finite.

Other conjectures

- For $r = 2n$ and $L = \mathcal{O}_C$: Exactly half of the rank n maximal isotropic subbundles of V belong to each component of the orthogonal Grassmannian bundle $\mathrm{OG}(n, V)$.
- For $L = \mathcal{O}_C$: The number of rank $n-1$ maximal subbundles in rank $2n-1$ is exactly half the number of rank n maximal isotropic subbundles for rank $2n$.

These are under investigation with D. Cheong and I. Choe.

Thank you and happy birthday Peter!