

Motivic mirror symmetry for Higgs bundles

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joint work with
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§1 Moduli spaces of Higgs bundles

C/k smooth projective connected alg. curve

Fix a rank n & degree d s.t. $(n,d) = 1$

Moduli spaces:

- $N_{n,d}$ moduli space of stable rank n degree d vector bundles on C

$T_{[E]}^* N_{n,d} \simeq \text{Ext}'(E, E)^* \simeq \text{Hom}(E, E \otimes \omega_C)$
 smooth projective variety of dim $n^2(g-1)+1$ def. theory Serre Duality \uparrow
 as $\text{Hom}(E, E) = k$ for E stable \uparrow Higgs fields

- $M_{n,d}$ moduli space of stable rank n degree d Higgs bundles on C
 $(E, \bar{\Phi}: E \rightarrow E \otimes \omega_C)$
 smooth q-proj variety $\dim M_{n,d} = 2 \dim N_{n,d}$

Geometric features:

* Over $k = \mathbb{C}$, M is a non-compact hyperkähler mfld (In general M is algebraic symplectic)

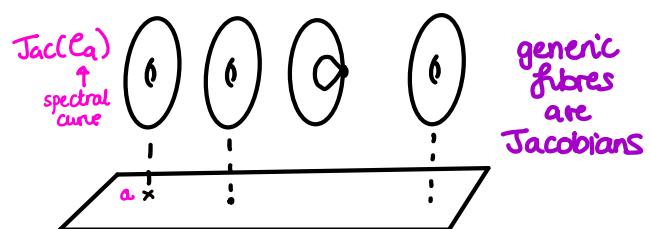
* Cohomologically M behaves like a compact mfld, due to the existence of a "semi-projective" scaling action: $G_m = \mathbb{R}^\times \curvearrowright M$
 $t \cdot [E, \bar{\Phi}] = [E, t\bar{\Phi}]$
 [Hitchin, Simpson, Hausel...]

* Hitchin fibration:

completely integrable Hamiltonian system

M
 $h \downarrow$
 affine space $\xrightarrow{\sim} \mathbb{A}$

$(E, \bar{\Phi})$
 \downarrow proper
 $\text{Char}(\bar{\Phi})$



* Versions for other reductive groups $G \neq \text{GL}_n \rightsquigarrow h_G: M_G \rightarrow \mathbb{A}_G$

- SL_n -Higgs bundles : for $L \in \text{Pic}^d(C)$

$$M_{n,L} = \{(E, \bar{\Phi}) \in M_{n,d} : \det(E) \simeq L, \text{Tr}(\bar{\Phi}) = 0\} \hookrightarrow M_{n,d}$$

$h_L \downarrow \Gamma := \text{Jac}(C)[n] \quad (n,d)=1 \Rightarrow M_{n,L} \text{ smooth} \quad \overset{\mathbb{C}}{\text{Jac}(C)}$

$\mathfrak{A}_{n,L}$

- PGL_n -Higgs bundles

$$\bar{M}_{n,d} := [M_{n,L} / \Gamma] \simeq [M_{n,d} / \Gamma^* \text{Jac}(C)] \text{ smooth orbifold (Deligne-Mumford stack)} \\ + \mu_n\text{-gerbe } \delta_L \quad \text{whose coarse m.space is singular}$$

Mirror symmetry for SL_n & PGL_n -Higgs moduli spaces ($k = \mathbb{C}$)

- Langlands dual groups have isomorphic Hitchin bases :

$$\begin{array}{ccc} SL\text{-Higgs} & M_{n,L} & \bar{M}_{n,d} \\ & h_L \searrow & \swarrow \bar{h} \\ & \mathfrak{A}_{n,L} & \end{array} \quad PGL\text{-Higgs}$$

- [Hausel-Thaddeus]: generic fibres of h_L & \bar{h} are dual abelian varieties

$$h_L^{-1}(a) \hookrightarrow \text{Prym}(e_a/\mathbb{C}) \xrightarrow{\sim} \text{Prym}(e_a/\mathbb{C}) / \Gamma \subset \bar{h}^{-1}(a) \\ \text{Prym}(e_a/\mathbb{C})^\vee$$

- Expected derived equivalence

$$D_{coh}^b(M_{n,L}) \simeq D_{coh}^b(\bar{M}_{n,d}, \delta_L) \text{ relative to } \mathfrak{A}_{n,L}$$

[Donagi-Pantev]: for $G \times^L G$ -Higgs bundles, there is such a derived equiv. over an open subset of the Hitchin base

- Topological mirror symmetry (Conjectured by Hausel & Thaddeus)

Proved by Groechenig-Wyss-Ziegler & also Maulik-Shen

$$(*) \quad H^*(M_{n,L}; \mathbb{C}) \simeq H_{orb}^*(\bar{M}_{n,d}, \delta_L; \mathbb{C}) \text{ as (pure) Hodge structures}$$

Orbifold cohomology twisted by the gerbe δ_L :

$$H_{orb}^*(\bar{M}_{n,d}, \delta_L) := \bigoplus_{\substack{\sigma \in \Gamma \\ \text{Jac}(C)[n]}} H^{*-2dr}(\mathcal{M}_\sigma)_{K_\sigma}(-d_\sigma) \quad \text{where } \begin{aligned} & \bullet \text{ dr} := \text{codim}(\mathfrak{A}_\sigma \hookrightarrow \mathfrak{A}_{n,L}) \\ & \bullet K_\sigma \leftrightarrow \sigma \\ & \bullet \hat{\Gamma} \simeq \Gamma \text{ Weil pairing} \end{aligned}$$

$\mathcal{M}_\sigma \hookrightarrow M_{n,L}$
 $\downarrow h_L$
 $\downarrow h_L$

$(\mathcal{M}_\sigma)^\sigma$
 $\overset{\mathbb{C}}{\Gamma}$

K_σ -isotypical piece
(for Γ -action)

Since $H^*(M_{n,L}) = \bigoplus_{K \in \Gamma} H^*(M_{n,L})_K$ (isotypical decomposition)

(*) follows from isos $H^*(M_{n,L})_{K_\sigma} \simeq H^{*-2d_\sigma}(M_\sigma)_{K_\sigma}(-d_\sigma)$ $\forall \sigma \in \Gamma$

- Maulik & Shen**
- construct this iso from an iso relative to the Hitchin base
 - use perverse sheaves + decomp. thm & vanishing cycles

Goal: prove a motivic refinement of topological mirror symmetry

motive: in the sense of Grothendieck

(realised by Voevodsky's triangulated category $DM(k, \mathbb{Q})$)

* encodes cohomology groups:

- ($k = \mathbb{C}$) singular cohomology + mixed Hodge structure
- ℓ -adic cohomology + Galois representation

* and algebraic cycles (Chow groups)

Main Theorem [H-Pépin Behalleur] "Motivic Mirror Symmetry"

Let $k = \bar{k}$ be a field with $\text{char}(k) = 0$ & let $\Lambda := \mathbb{Q}(\zeta_n)^{\leftarrow \text{primitive } n^{\text{th}} \text{ root of unity}}$

In Voevodsky's category $DM(k, \Lambda)$ of motives over k with Λ -coeffs
there is an isomorphism of (twisted orbifold) motives:

$$M(M_{n,L}) \simeq M_{\text{orb}}(\bar{M}_{n,d}, \delta_L)$$

Corollary: Mirror symmetry for (twisted orbifold) Chow gps with Λ -coeffs

§2 Motives

Grothendieck: envisaged motives as a universal coh theory.
for k -varieties

Voevodsky: There is a category $DM(k, \mathbb{Q})$ of mixed motives / k
with \mathbb{Q} -coefficients together with a functor

$$\begin{aligned} M: \text{Var}_k &\longrightarrow DM(k, \mathbb{Q}) \\ X &\longmapsto M(X) \end{aligned}$$

Realising part of Grothendieck's vision.

Properties

- * Künneth isomorphism $M(X \times Y) \cong M(X) \otimes M(Y)$
 - * A^1 -homotopy invariance: $E \xrightarrow{\text{vector bundle}} X \rightsquigarrow M(E) \cong M(X)$
 - * Projective bundle formula: $M(\mathbb{P}(E)) \cong M(X) \otimes M(\mathbb{P}^{n-1})$
- $$M(\mathbb{P}^n) = \bigoplus_{i=0}^n \mathbb{Q}\{i\} \xleftarrow{\text{Tate twists}} n = rk(E)$$

- * Gysin triangles: for $Z \hookrightarrow X$ both smooth k-varieties
 $\text{codim } c$
- $$M(X \setminus Z) \rightarrow M(X) \rightarrow M(Z) \{c\} \xrightarrow{+1}$$

- * Chow gps: X smooth k-variety

$$CH^i(X)_{\mathbb{Q}} \simeq \underset{DM}{\text{Hom}}(M(X), \mathbb{Q}\{i\})$$

- * Realisation functors:

Betti / de Rham / ℓ -adic cohomology factor via $M: \text{Var}_k \rightarrow DM$

$MHS \xrightarrow{+} Galois \text{ rep.} \xrightarrow{+}$

Eg. For $k \hookrightarrow \mathbb{C}$, \exists Betti realisation $R_B: DM(k, \mathbb{Q}) \rightarrow D(\mathbb{Q}\text{-vect})$

- * Relative motives & six operations formalism: for varieties over S , there are categories of relative motives $DM(S, \mathbb{Q})$

$f: T \rightarrow S$ sep. f-type $\rightsquigarrow f^*: DM(S, \mathbb{Q}) \rightleftarrows DM(T, \mathbb{Q}): f_*$
as well as $f_!, f^!, \otimes, \underline{\text{Hom}}$

§ 3 Outline of the proof

Part A: lift the construction of Maulik-Shen to DM

• Construct a morphism $\beta: (h_L * \Lambda)_{K_S} \rightarrow i_{g_*} (h_{g_*} * \Lambda)_{K_T} \{ -d_S \}$ in $DM(\mathbb{A}_L, \Lambda)$

whose Betti realisation $R_B(\beta)$ is the isomorphism of Maulik & Shen

• Pushforward β along the structure map $\mathbb{A}_L \rightarrow \text{Spec } k$ & dualise

$\rightsquigarrow \alpha: M(M_S)_{K_S} \{d_S\} \rightarrow M(M_{h,L})_{K_S}$ in $DM(k, \Lambda)$

Part B: Prove that α is an isomorphism via a conservativity argument

- Prove that both sides are abelian motives (following geom. ideas of García-Prada, Heinloth & Schmitt)

$$\begin{aligned} \mathrm{DM}_c^{\mathrm{ab}}(k, \mathbb{Q}) &:= \langle M(A) : A \text{ abelian variety} \rangle \\ &= \langle M(C) : C \text{ smooth proj. curve} \rangle \end{aligned}$$

- Apply Wildeshaus' conservativity thm for Betti realisation on abelian motives:

Thm [Wildeshaus] for $k \hookrightarrow \mathbb{C}$, the Betti realisation

$$R_B : \mathrm{DM}_c^{\mathrm{ab}}(k, \mathbb{Q}) \rightarrow D(\mathbb{Q}\text{-vect}) \text{ is conservative.}$$

§3 Part A: Construction of β

- ① First guess: construct β_{naive} using a motivic endoscopic correspondence

Maulik-Shen construct a cohomological correspondence based on work of Ngô & Yun

$$\rightsquigarrow \beta_{\text{naive}} : (h_{L^+} \wedge)_{K_S} \rightarrow i_{\gamma^+}^* (h_{\gamma^+} \wedge)_{K_S} \{ -d_\gamma \} \text{ in } \mathrm{DM}(S_L, \wedge)$$

Yun: The corresponding cohomological correspondence is an isomorphism on a dense open in the Hitchin base S_L

Problem: It's unclear if β_{naive} is an iso over the full Hitchin base.

- ② Consider D-twisted Higgs bundles
 $\Leftrightarrow (E, \bar{\Phi} : E \rightarrow E \otimes \mathcal{O}_C(D))$

$\rightsquigarrow h_L^D : M_{n,L}^D \xrightarrow{\text{proper}} S_L^D$
no longer \Rightarrow alg. symplectic
Topology of h_L^D is simpler for $\deg D > 2g-2$

using endoscopic corr. of Yun & Ngô:

$$\rightsquigarrow \beta_{\text{naive}}^D : (h_{L^+}^D \wedge)_{K_S} \rightarrow i_{\gamma^+}^* (h_{\gamma^+}^D \wedge)_{K_S} \{ -d_\gamma^D \}$$

Thm [Maulik-Shen] For D of even degree & $\deg(D) > 2g-2$

$R_B(\beta_{\text{naive}}^D)$ is an isomorphism

③ Use vanishing cycles to pass from D_{tp} to D (& Construct β^D from $\beta^{D+P}_{\text{naive}}$) following Maulik-Shen

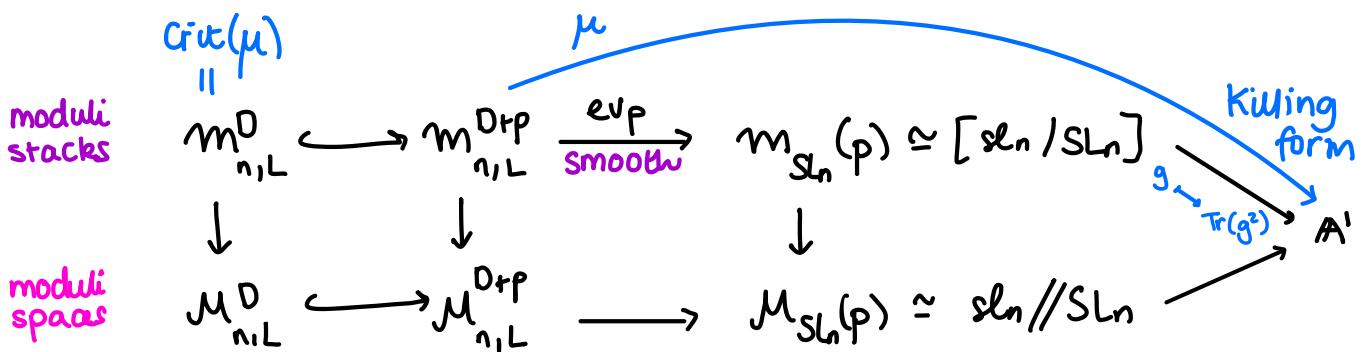
for $D = K_C$, we need to do this twice $K_C \hookrightarrow K_{C+P} \hookrightarrow K_{C+P+Q}$

Involves:

- extending Ayoub's motivic vanishing cycles to stacks
- describing motivic vanishing cycles for quadratic forms.

To pass from D_{tp} to D , M-S restrict to SL-Higgs bundles over the point p :

\uparrow
trace free matrices



vanishing cycles
 $\rightsquigarrow \beta^D : (h_{L^*}^D \wedge)_{K_S} \rightarrow i_{\infty}^D \ast (h_{\infty}^D \wedge)_{K_S} \{ -d_{\infty}^D \}$ for $\deg(D) > 2g-2$
 and also $D = K_C$

s.t. $R_B(\beta^D)$ is the Maulik-Shen isomorphism.

§4 Part B: The motives involved are abelian

Thm 1 [H-Pepin Hebecker '19]

The motive of the GL-Higgs moduli space is generated by $M(C)$:

For $(n, d) = 1$, we have $M(M_{n,d}) \in \langle M(C) \rangle \subset DM^{\text{ab}}(k, \mathbb{Q})$.

Our proof uses geom. ideas of García-Prada, Heinloth & Schmitt

Thm 2 [H-Pepin Lehalleur '22]

Assume $C(k) \neq \emptyset$ & degree $D \geq 2g-2$. For $(n,d) = 1$, we have

$$i) M(M_{n,d}^D) \in \langle M(C) \rangle \subset DM^{ab}(k, \mathbb{Q})$$

$$ii) M(M_{n,L}^D) \in \langle M(\tilde{C}) : \tilde{C} \xrightarrow{\text{certain finite \'etale covers}} C \rangle \subset DM^{ab}(k, \mathbb{Q})$$

On the level of cohomology Hitchin observed this for rank $n=2$.

⚠ $M(M_{n,L}) \notin \langle M(C) \rangle$ in general [Fu-H.-Pepin Lehalleur]

Whereas for vector bundle moduli: $M(N_{n,d})$ and $M(N_{n,L}) \in \langle M(C) \rangle$.

In fact $M(N_{n,d}) \simeq M(N_{n,L}) \otimes M(\text{Jac } C)$ [Fu-H.-Pepin Lehalleur]

motivic version[↑] of cohomological thm
of Harder & Narasimhan.

Sketch proof of Thm 1:

① Hitchin's scaling action $\mathbb{G}_m \curvearrowright M_{n,d}$ $t \cdot [E, \Phi] = [E, t\Phi]$

↗ deformation retract to fixed points (Bialynicki-Birula)

↑
chains of vector bundles

$$\mathbb{G}_m \curvearrowright E \rightsquigarrow E = \bigoplus E_i$$

$$F_0 \rightarrow F_1 \rightarrow \dots \rightarrow F_r$$

$$F_i = E_i \otimes \omega_C^{\otimes i}$$

$$M_{n,d}$$

↓ flow
 $t \rightarrow 0$

$$(M_{n,d})^{\mathbb{G}_m} = \coprod_{(\underline{m}, \underline{e})} Ch_{\underline{m}, \underline{e}}^{\alpha_{H-SS}}$$

finitely many tuples
of ranks & degrees

Higgs stability param. for chains

← Chain moduli spaces
(smooth projective varieties)

↗ motivic BB decomposition \Rightarrow suffices to describe motives of
chain moduli spaces

② Use wall-crossing on space of chain stability parameters to reduce to injective chains of constant rank $F_0 \hookrightarrow F_1 \hookrightarrow \dots \hookrightarrow F_r$

Based on [García-Prada, Heinloth, Schmitt]: for virtual motive in $\widehat{K}_0(\text{Var})$

* Deformation theory for α -ss chains best understood for $\alpha \in \Delta_r^\circ \subset \mathbb{R}^{r+1}$ cone
[Álvarez-Gavela, García-Prada, Schmitt]

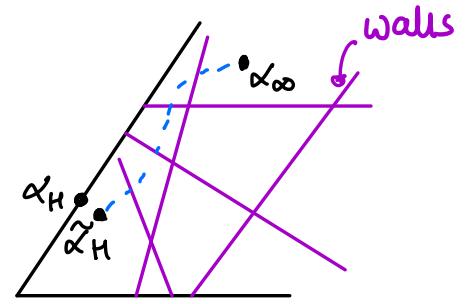
* Thm [García-Prada, Heinloth, Schmitt]

\exists path $\{\alpha_t\}_{t \geq 0}$ in Δ_r° from $\tilde{\alpha}_H$

to $\alpha_{\infty} = \alpha_t$ for $t > 0$ s.t.

i) If \underline{m} is non-constant, $\text{Ch}_{\underline{m}, e}^{\alpha_{\infty}-\text{ss}} = \emptyset$

ii) If \underline{m} is constant, $\text{Ch}_{\underline{m}, e}^{\alpha_{\infty}-\text{ss}} \subset \text{Ch}_{\underline{m}, e}^{\text{inj}} \leftarrow \text{stack of injective chains}$



* Gysin triangles for Harder-Narasimhan strata at each wall-crossing + HN-recursion.

③ Explicit formula for the motive of the stack of injective chains

Inspired by work of Heinloth & Laumon (on cohomology) using stacks of Hecke modifications & motivic descriptions of small maps (de Cataldo - Miglionini)

Thm [H.-Pépin Lehalleur]

For $\underline{m} = (m_0, \dots, m_r)$ constant,

$$M(\text{Ch}_{\underline{m}, e}^{\text{inj}}) \cong \bigotimes_{i=1}^r M(\text{Sym}^{e_i - e_{i-1}}(C \times \mathbb{P}^{n_i})) \otimes M(\text{Bun}_{m_i, e_i}) \text{ in } \text{DM}(k, \mathbb{Q}).$$

Proof: Based on ideas of Heinloth, Laumon and de Cataldo & Miglionini

$$\begin{array}{ccccc}
\mathcal{E}h_{m,e}^{\text{inj}} & \xrightarrow{\text{gr (smooth)}} & \prod_{i=1}^r \text{Coh}_{0, \ell_i} \times \text{Bun}_{m_r, e_r} & \xrightarrow{\text{supp}} & \prod_{i=1}^r C^{(\ell_i)} \times \text{Bun}_{m_r, e_r} \\
\uparrow \text{small [Heinloth]} & \leftarrow \text{small map [Haunon]} & & & \uparrow \\
\tilde{\mathcal{E}h}_{m,e}^{\text{inj}} & \xrightarrow{\tilde{\text{gr}}} & \prod_{i=1}^r \tilde{\text{Coh}}_{0, \ell_i} \times \text{Bun}_{m_r, e_r} & \xrightarrow{\sim \text{supp}} & \prod_{i=1}^r C^{\ell_i} \times \text{Bun}_{m_r, e_r} \\
\uparrow \sum_{i=1}^r \{ F_0 \subset F_0' \subset \dots \subset F_0^{\ell_1} = F_1 \subset \dots \subset F_{r-1}^{\ell_r} = F_r \} & & \uparrow \{ 0 \subset T_1 \subset \dots \subset T_{\ell_i} : T_j \in \text{Coh}_{0,j} \} & & \\
& & \sum_{i=1}^r \ell_i - \text{iterated } \mathbb{P}^{m-1} - \text{bdle} & \xleftarrow{\quad \quad \quad} & (\text{A Hecke modification of a rk } m) \\
& & & & \text{bdle } F \text{ at } p \in C(k) \Leftrightarrow F_p \rightarrow k
\end{array}$$

$$\Rightarrow M(\tilde{\mathcal{E}h}_{m,e}^{\text{inj}}) \simeq M(\text{Bun}_{m_r, e_r}) \times \bigotimes_{i=1}^r M(C \times \mathbb{P}^{m-1})^{\otimes \ell_i}$$

Moreover, these small maps are torsors under $\prod_{i=1}^r S_{\ell_i}$ on a dense open
 where $\forall i \text{ supp}(F_i/F_{i-1})$
 consists of ℓ_i distinct pts

Following ideas of de Cataldo & Migliorini:

$$M(\mathcal{E}h_{m,e}^{\text{inj}}) = M(\tilde{\mathcal{E}h}_{m,e}^{\text{inj}})^{\prod_{i=1}^r S_{\ell_i}} \simeq M(\text{Bun}_{m_r, e_r}) \otimes \bigotimes_{i=1}^r M(\text{Sym}^{\ell_i}(C \times \mathbb{P}^{m-1}))$$

④ Explicit formula for the stack Bun of vector bundles:

Thm [H-Pepin Lehalleur '18] Assume $C(k) \neq \emptyset$. For any m, e :

$$M(\text{Bun}_{m,e}) \simeq M(\text{Jac } C) \otimes M(BG_m) \otimes \bigotimes_{i=1}^{m-1} \underbrace{\mathcal{Z}(C, \mathbb{Q} \oplus i)}_{\text{II}}$$

In particular, $M(\text{Bun}_{m,e}) \in \langle\langle M(C) \rangle\rangle$.

$$\bigoplus_{j \geq 0} M(\text{Sym}^j(C)) \{ij\}$$

Rmk: The proof of Thm 2 (on the motive of the **SL-Higgs** m.space) involves modifying the above approach: Steps ① & ② are the same, but for Step ③, we consider injective chains of **fixed total determinant**