VBAC 2022:
On the occasion of Peter Newstead's 80th Birthday

e-Polynomials of character varieties of surface groups (I)

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28th July 2022, Warwick

Introduction

Σ_g compact Riemann surface of genus g

$M(r,d)$ moduli space of stable vector bundles of rank r and degree d on Σ_g

Mumford 1962: $M(r,d)$ has a natural structure of an algebraic variety

Narasimhan-Seshadri 1964 & 1965: topological type of $M(r,d)$ depends on g

Topology Vol. 6, pp.241–262. Pergamon Press, 1967. Printed in Great Britain

TOPOLOGICAL PROPERTIES OF SOME SPACES OF STABLE BUNDLES

P. E. NEWSTEAD

(*Received* 9 *September* 1966)

OUR OBJECT in this paper is to study the topology of certain spaces which arise in connection with the classification of holomorphic vector bundles. Mumford [5, 6] has introduced the concept of a stable bundle over a compact Riemann surface X and has proved that the set of stable bundles of fixed rank and degree over X has a natural structure of non-singular quasi-projective algebraic variety (see [10]). It is not difficult to deduce that the set of such bundles of fixed determinant has a similar structure, and that the set of projective bundles arising from such bundles has a structure of quasi-projective variety (possibly with singularities). Narasimhan and Seshadri [7, 8] have shown that the topological type of these varieties depends only on the genus of X.

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My thanks are due to M. F. Atiyah, M. S. Narasimhan and C. S. Seshadri for many useful discussions during the preparation of this paper; also to Miss Ann Garfield, who typed two versions of it.

We consider the differentiable map

$$f_g : SU(2)^{2g} \to SU(2)$$

defined by

$$(A_1, B_1, \ldots, A_g, B_g) \to \prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1},$$

and in particular the subspace $S_0^{(g)} = f_g^{-1}(-I)$ of $SU(2)^{2g}$. Now there is a differentiable action of $SU(2)$ on $SU(2)^{2g}$ defined by the formula:

$$T.(A_1, B_1, \ldots, A_g, B_g) = (TA_1T^{-1}, TB_1T^{-1}, \ldots, TA_gT^{-1}, TB_gT^{-1});$$

Introduction

$\Gamma = < x_1, \ldots, x_k \mid r_1, \ldots, r_s >$ finitely generated group

G complex algebraic group

Representation variety

$$\mathscr{X}(\Gamma, G) = \mathrm{Hom}(\Gamma, G) = \{(A_1, \ldots, A_k) \in G^k \mid r_1, \ldots r_s \text{ satisfied}\}$$

Example:

$$\Gamma = \pi_1(\Sigma_1) = < a, b \mid aba^1b^{-1} = 1 > \quad G = \mathrm{GL}_2(\mathbb{C})$$

$$\mathscr{X}(\Gamma, G) = < (A, B) \in \mathrm{GL}_2(C)^2 \mid ABA^{-1}B^{-1} = 1 >$$

Introduction

G acts on $\mathscr{X}(\Gamma, G)$ by conjugation

Character variety

$$\mathfrak{M}(\Gamma, G) := \mathscr{X}(\Gamma, G)//G$$

Example:

$$\mathfrak{M}(< a >, \mathrm{SL}_2(\mathbb{C})) = \mathrm{SL}_2(\mathbb{C})//\mathrm{SL}_2(\mathbb{C})$$

For $G = \mathrm{GL}_r(\mathbb{C}), \mathrm{SL}_r(\mathbb{C}), \mathrm{PSL}_r(\mathbb{C})$ $\mathfrak{M}(\Gamma, G)$ can be parametrised by characters

Culler and Shalen (1983): Character $\chi_\rho : \Gamma \longrightarrow k$; $\chi_\rho(g) = \mathrm{Tr}(\rho(g))$, so that $\rho_1, \rho_2 : \Gamma \longrightarrow \mathrm{GL}(V_k)$ isomorphic $\Longrightarrow \chi_{\rho_1} = \chi_{\rho_2}$

Therefore $\mathrm{Hom}^{irr}(\Gamma, \mathrm{GL}(V))/G$ is the space of characters

Introduction

Narasimhan-Seshadri 1965: $\mathfrak{M}(\pi_1(\Sigma_g), U(r)) \cong M(r,0)$

$$\mathfrak{M}_I^g(U(r)) := \{(A_1, B_1, \cdots, A_g, B_g) \in U(r)^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = I\} // U(r)$$

For $G = \mathrm{GL}_r(\mathbb{C}), \mathrm{SL}_r(\mathbb{C})$

$\mathfrak{M}_I^g(G) \cong G$-local systems $(\deg = 0)$

$$\mathfrak{M}^g_{e^{\frac{2\pi i d}{r}} I}(G) = \{(A_1, B_1, \cdots, A_g, B_g) \in G^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = e^{\frac{2\pi i d}{r}} I\} // G \; G-\text{local}$$

systems such that (E, ∇) residue$(\nabla)=-\dfrac{d}{r}$

$$\mathfrak{R}^g_{e^{\frac{2\pi i d}{r}} I} := \{(A_1, B_1, \cdots, A_g, B_g) \in G^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = e^{\frac{2\pi i d}{r}} I\}$$

Introduction

If G is abelian, any X

$$\mathrm{Hom}(\pi_1(X), G) \cong \mathrm{Hom}\left(\frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}, G\right) \cong H_1(X, G) \cong H^1(X, G)$$

When $G = \mathbb{R}$ (X e.g. differential manifold) we get $H^1(X, \mathbb{R}) \cong H^1_{dR}(X, \mathbb{R})$

Moreover $X = \Sigma_g$ Riemann surface gives $H^1_{dR}(\Sigma_g, \mathbb{R}) \cong \mathscr{H}^1(\Sigma_g)$

Hence we have a correspondence:

$\mathrm{Hom}(\pi_1(\Sigma_g), \mathbb{R})$

$H^1_{dR}(\Sigma_g)$

$\mathscr{H}^1(\Sigma_g)$

Introduction

Any reductive algebraic group G

<u>Theorems by Corlette, Donaldson, Hitchin and Simpson</u>: (1980s)

$$\mathscr{M}^d_{Higgs}(G) \cong \mathscr{M}^d_{Betti}(G) \cong \mathscr{M}^d_{de\,Rham}(G)$$

G-Local systems

Representations of the fundamental group into G

Moduli space of G-Higgs bundles

We shall deal also with non-reductive Lie groups!

Hodge structures

Weight k Hodge structure:

H finite dimensional complex vector space with real structure

$$H = \bigoplus_{k=p+q} H^{p,q},\ H^{p,q} = \overline{H^{q,p}}$$

$$H = F^0 \supset F^1 \supset F^2 \supset \cdots,\ \ F^p = \bigoplus_{s \geq p} H^{s,k-s}$$ Hodge

(descending) filtration

$$\mathrm{Gr}_F^p(H) = F^p / F^{p+1} = H^{p,q}$$

Thm (Deligne) For any compact Kähler variety Z its cohomology groups $H^k(Z)$ have a weight k pure Hodge structure

Hodge structures

Mixed Hodge structure:

H finite dimensional vector space with real structure

Weight ascending filtration

$$\cdots W_{k-1} \subset W_k \subset \cdots \subset H$$

Hodge descending filtration $F^\bullet$ inducing a pure Hodge structure of weight k on each $\mathrm{Gr}_k^W := W_k / W_{k-1}$

$$H^{p,q} = \mathrm{Gr}_F^p \mathrm{Gr}_{p+q}^W \quad h^{p,q} := \dim H^{p,q}$$

Thm (Deligne) For any Z quasi-projective variety, $H^k(Z)$ and $H_c^k(Z)$ have mixed Hodge structures

$$H^k(Z) = \bigoplus_{p,q} H_c^{k;p,q}(Z) \qquad H^{k;p,q}(Z) = \mathrm{Gr}_F^p \mathrm{Gr}_{q+p}^W (H_c^k(Z))$$

Invariant polynomials

Poincaré polynomial $P(Z)(t) := \sum_k b_k t^k \, , \, b_k = \dim H^k(Z)$

Hodge polynomial $H(Z)(u,v) := \sum_{p,q} h^{p,q}(Z) u^p v^q$

Mixed polynomial $H(Z)(u,v,t) := \sum_{k,p,q} h_c^{k,p,q}(Z) t^k u^p v^q$

e-polynomial $e(Z)(u,v) := \sum_{p,q} \sum_k (-1)^k \, h_c^{k;p,q}(Z) u^p v^q \in \mathbb{Z}[u,v]$

$P(Z)(t) = e(Z)(t,t)$

e-polynomial

$Z = \sqcup_i Z_i \quad e(Z) = \sum_i e(Z_i)$

$Z = F \times B$ Küneth $e(Z) = e(F)e(B)$

$F \longrightarrow Z \longrightarrow B$ fibration in Zariski topology (or principal bundle)
$e(Z) = e(F)e(B)$

Examples:

$e(\mathbb{C}^n) = q^n$

$e(\mathbb{C}^\star) = q^n - 1$

$e(\mathbb{C}\mathbb{P}^n) = q^n + q^{n-1} + \cdots + q + 1$

e-polynomial

e : KVar$_k \longrightarrow \mathbb{Z}[u,v]$ ring homomorphism. In particular
$e([k]) = uv = q$

Z "balanced" : $h_c^{k;p,q}(Z) = 0$ whenever $p \neq q$

Moreover, if $[Z]$ lies in the subring generated by $[k]$ the virtual class extends the e-polynomial

We will get polynomials $e(q)$ as our invariants for our character varieties.

This is functorial and allows the construction of a TQFT

e-polynomial

$$e([\mathsf{GL}_2(\mathbb{C})]) = q^4 - q^3 - q^2 + q$$

$\mathsf{GL}_2(\mathbb{C}) \longrightarrow \mathbb{C}^2\backslash 0$ loc. trivial (Zariski top.) $A \mapsto Ae_1$

$$e([GL_2(\mathbb{C})]) = e([\mathbb{C}^2\backslash\{(0,0\}])e([\mathbb{C}^2 - \mathbb{C}]) = (q^2 - 1)(q^2 - q)$$

$$e([\mathsf{PGL}_2(\mathbb{C})]) = q^3 - q$$

$$e([\mathsf{SL}_2(\mathbb{C})]) = q^3 - q$$

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathsf{SL}_2(\mathbb{C}) \longrightarrow \mathbb{C}^2\backslash 0 \longrightarrow 0$$

$$e([\mathsf{SL}_2(\mathbb{C})]) = e([\mathbb{C}])e([\mathbb{C}^2\backslash 0]) = q(q^2 - 1)$$

e-polynomial

Example

$e([\mathscr{X}(\pi_1(\Sigma_g), \mathsf{GL}_1(\mathbb{C}))]) = e([\mathsf{GL}_1(\mathbb{C})]^{2g}) = (q-1)^{2g}$ because:

$$\mathfrak{R}_I^g(\mathsf{GL}_1(\mathbb{C})) = < A_1, B_1, \ldots, A_g, B_g \mid \prod_{i=1}^{g} [A_i, B_i] = 1 >$$

$A_i, B_i \in \mathsf{GL}_1(\mathbb{C})$

Consider

$$h : (A_1, B_1, \ldots, A_g, B_g) \in \mathsf{GL}_1(\mathbb{C})^{2g} \longrightarrow \prod_{i=1}^{g} [A_i, B_i] \in \mathsf{GL}_1(\mathbb{C})$$

$$\mathfrak{R}_I^g(\mathsf{GL}_1(\mathbb{C})) = h^{-1}(I)$$

e-polynomial

Goal $e(\mathfrak{R}^g_{C_0}(\mathrm{SL}_2(\mathbb{C}))$

$f : \mathrm{SL}_2(\mathbb{C})^{2g} \longrightarrow \mathrm{SL}_2(\mathbb{C})$ commutator map

$\mathrm{SL}_2(\mathbb{C}) = I \sqcup -I \sqcup [J^+] \sqcup [J^-] \sqcup C_\lambda$ where $C_\lambda = \mathrm{diag}(\lambda, \lambda^{-1})$ $\lambda \in \mathbb{C}^\star \backslash \pm 1$

$$J^+ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \; J^- = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

Notice $e(\pm I) = 1 \quad e([J^\pm]) = q^2 - 1 \quad e([C_\lambda]) = q^3 - 2q^2 - q$

$$e(\mathrm{SL}_2(\mathbb{C})) = 1 + 1 + 2(q^2 - 1) + q^3 - 2q^2 - q = q^3 - q$$

$\mathscr{C}_i \in \{I, -I, [J^+], [J^-], [C_\lambda]\}$

$\mathrm{SL}_2(\mathbb{C})^{2g} = \sqcup_i f^{-1}(C_i)$

e-polynomial

To compute $e([C_\lambda])$ we need

Proposition: Let X algebraic variety with a $\mathbb{Z}_2$ action, B smooth and irreducible, F variety

$$\begin{tikzcd}
 & F \arrow[dl] \arrow[dr] & \\
X \arrow[r] \arrow[d] & X/\mathbb{Z}_2 \arrow[d] \\
B \arrow[r] & B/\mathbb{Z}_2
\end{tikzcd}$$

Then $e(X)^+ = e(X/\mathbb{Z}_2) = e(F)^+ e(B)^+ + e(F)^- e(B)^-$

Where $e(X)^- = e(X) - e(X)^+$

e-polynomial

$X_0 = f^{-1}([I])$ $\quad X_1 = f^{-1}([-I])$ $\quad X_2 = f^{-1}([J^+])$ $\quad X_3 = f^{-1}([J^-])$
$X_4 = f^{-1}([C_\lambda])$

Also define:

$$\overline{X_0} = f^{-1}(I) = X_0 \qquad \overline{X_1} = f^{-1}(-I) = X_1$$

$$\overline{X_2} = f^{-1}(J^+) \quad \overline{X_3} = f^{-1}(J^-) \quad \overline{X_4} = f^{-1}(C_\lambda)$$

$$e(X_0) = e(\overline{X_0}) = q^4 + 4q^3 - q^2 - 4q$$

$\overline{X_0} = \{(A, B) \mid [A, B] = I\}$ decompose it as

(1) $\begin{cases} A = \pm I \text{ and } B \in \mathrm{SL}_2(\mathbb{C}) \text{ s.t. } ABA^{-1}B^{-1} = BB^{-1} = I \text{ hence } \mathrm{SL}_2(\mathbb{C}) \sqcup \mathrm{SL}_2(\mathbb{C}) \\ B = \pm I \text{ and } A \in \mathrm{SL}_2(\mathbb{C}) \text{ s.t. } ABA^{-1}B^{-1} = AA^{-1} = I \text{ hence } \mathrm{SL}_2(\mathbb{C}) \sqcup \mathrm{SL}_2(\mathbb{C}) \end{cases}$

$$e((1)) = e(\mathrm{SL}_2(\mathbb{C}) \sqcup \mathrm{SL}_2(\mathbb{C}) \sqcup \mathrm{SL}_2(\mathbb{C}) \sqcup \mathrm{SL}_2(\mathbb{C}) \backslash \pm I \times \pm I) = 4(q^3 - 1) - 4$$

e-polynomial

$$e(X_0) = e(\overline{X_0}) = q^4 + 4q^3 - q^2 - 4q$$

$$\overline{X_0} = \{(A,B) \mid [A,B] = I\} = (1) \sqcup (2) \sqcup (3)$$

(2) $A, B \in [J^{\pm}]$ Jordan form in same basis

$$e((2)) = 4(q-1)e(\mathrm{PGL}_2(\mathbb{C}))/U) = 4(q-1)(q^2-1)$$

(3) $A, B \in [C_\lambda]$ diagonalisable same basis

$$e((3)) = (q-2)^2q^2 - 2(q-2)q + q^2$$

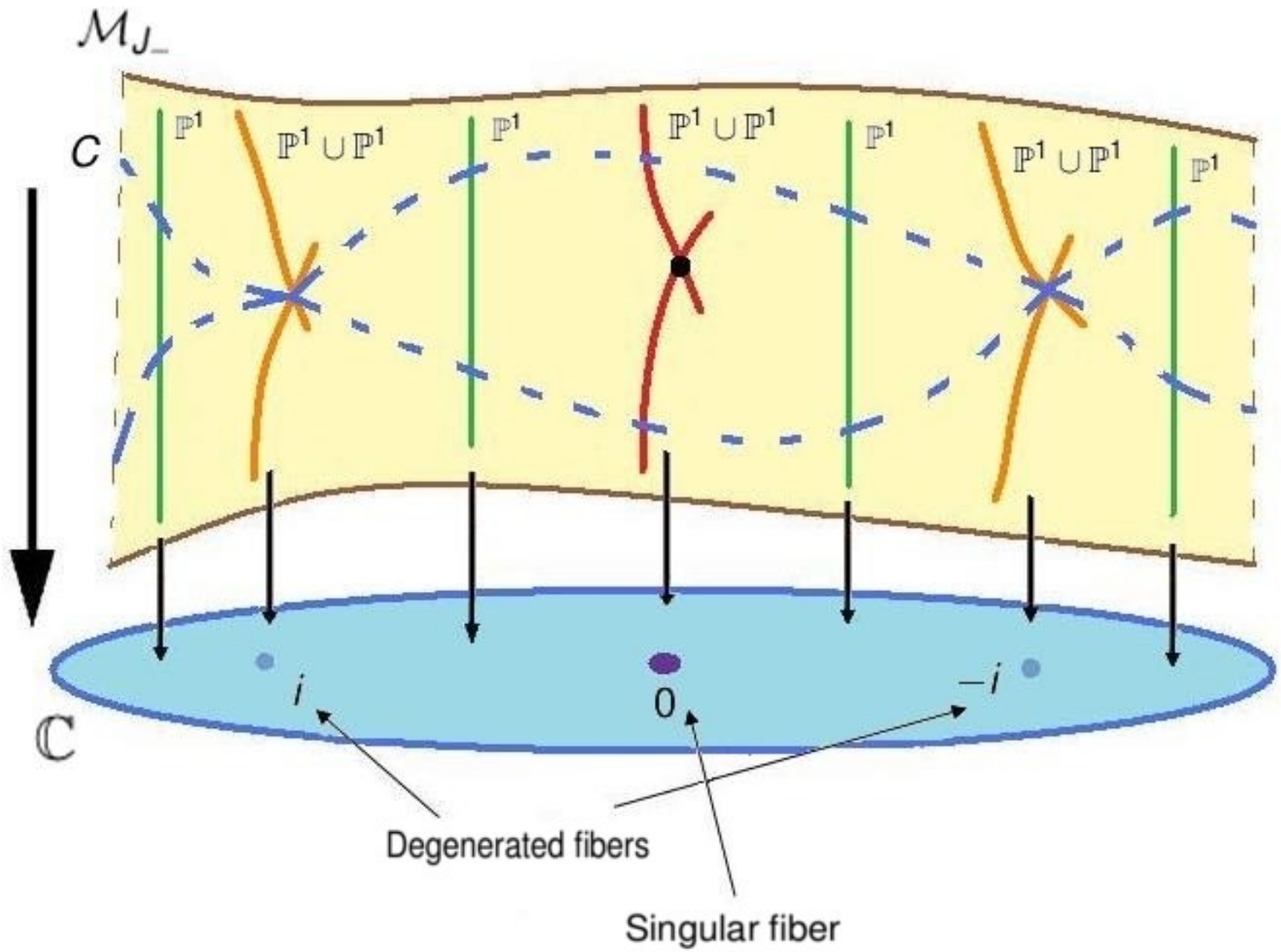
equivariant e-polynomial to consider the action of $\mathbb{Z}_2$ interchanging eigenvectors

$$e(\overline{X_1}) = q^3 - q = e(X_1)$$

$$e(\overline{X_2}) = q^3 - 2q^2 - 3q \qquad e(X_2) = e(\mathrm{GL}_2(\mathbb{C}/U)e(\overline{X_2}) = q^5 - 2q^4 - 4q^3 + 2q^2 + 3q$$

$$e(\overline{X_3}) = q^3 + 3q^2 \qquad e(X_3) = q^5 + 3q^4 - q^3 - 3q^2$$

$$e(\overline{X_4}) = q^4 - 3q^3 - 6q^2 + 5q + 3 \qquad e(X_4) = q^6 - 2q^5 - 4q^4 + 3q^2 + 2q$$



By Angel González-Prieto

Applications

$e(\mathfrak{M}^1_I(\mathrm{SL}_2(\mathbb{C}))) = q^2 + 1$ explicit description
$h^{4,2,2}(\mathfrak{M}^1_I(\mathrm{SL}_2(\mathbb{C}))) = h^{2,0,0}(\mathfrak{M}^1_I(\mathrm{SL}_2(\mathbb{C}))) = 1$ and

$$H_c(\mathfrak{M}^1_I(\mathrm{SL}_2(\mathbb{C})))(q,t) = q^2t^4 + t^2$$

$e(\mathfrak{M}^1_{-I}(\mathrm{SL}_2(\mathbb{C}))) = 1$

$e(\mathfrak{M}^1_{J+}(\mathrm{SL}_2(\mathbb{C}))) = q^2 - 2q - 3$ explicit description
$h_c^{4,2,2}(\mathfrak{M}^1_{J+}(\mathrm{SL}_2(\mathbb{C}))) = 1$ $h_c^{3,1,1}(\mathfrak{M}^1_I(\mathrm{SL}_2(\mathbb{C}))) = 2$
$h^{2,0,0}(\mathfrak{M}^1_{J+}(\mathrm{SL}_2(\mathbb{C}))) = 1$ $h^{1,0,0}(\mathfrak{M}^1_{J+}(\mathrm{SL}_2(\mathbb{C}))) = 4$

$P_c(\mathfrak{M}^1_{J+}(\mathrm{SL}_2(\mathbb{C}))) = t^4 + 2t^3 + t^2 + 4t$
$H_c(\mathfrak{M}^1_{J+}(\mathrm{SL}_2(\mathbb{C}))) = q^2t^4 + 2qt^3 + t^2 + 4t$

Applications

$e(\mathfrak{M}^1_{J^-}(\mathrm{SL}_2(\mathbb{C})) = q^2 + 3q$ explicit description $h_c^{4,2,2} = 1 = h_c^{1,0,0}$

$P_c(\mathfrak{M}^1_{J^-}(\mathrm{SL}_2(\mathbb{C})))(t) = t^4 + t^3 + 5t^2 + t$

	1	5	1	1	
					0
					3
					1

	1	5	1	1	
					0
					3
					1

Applications

$$e(\mathfrak{M}^1_{C_\lambda}(\mathrm{SL})_2(\mathbb{C}))) = q^2 + 4q + 1$$

Boden-Yokogawa: $P_c(\mathfrak{M}^1_{C_\lambda}(\mathrm{SL}_2(\mathbb{C}))) = t^4 + 5t^2$

	0	5	0	1	
					1
					4
					1

$$H_c(\mathfrak{M}^1_{C_\lambda}(\mathrm{SL}_2(\mathbb{C}))) = q^2t^4 + qt^2 + t^2$$

Applications

Parabolic structure:

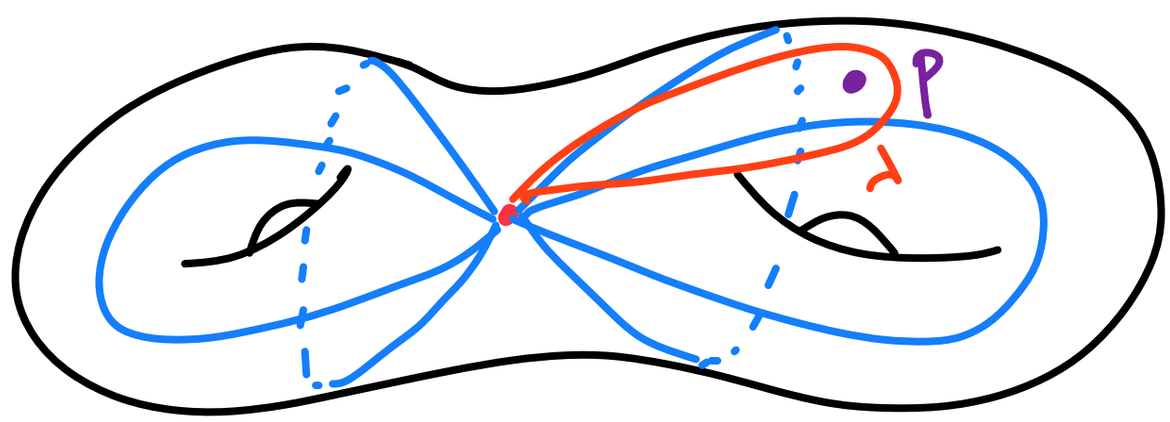
$$D := p_1 + p_2 + \cdots + p_s$$

$\{\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_s\}$ conjugacy clases in G (holonomies)

$$\mathfrak{R}^g_{\mathscr{C}_1,\ldots,\mathscr{C}_s}(G) := \{(A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n) \in G^{2g+s} \mid \prod_{i=1}^{2g} [A_i, B_i] C_1 \ldots C_s = 1\}$$

Parabolic representation variety

$\mathfrak{M}^g_{\mathscr{C}_1,\ldots,\mathscr{C}_s}(G) := \mathfrak{R}^g_{\mathscr{C}_1,\ldots,\mathscr{C}_s}(G)//G$ Parabolic character variety



Introduction

Facultad de Ciencias MATEMÁTICAS

Any reductive algebraic group G

C. Simpson Harmonic bundles on noncompact curves. *J. Amer. Math. Soc.* 3 (1990), no. 3, 713–770

$$\mathscr{M}_{PHiggs}(G) \cong \mathscr{M}^{p}_{Betti}(G) \cong \mathscr{M}^{p}_{de\,Rham}(G)$$

G-Local systems

Representations of the fundamental group into G

Moduli space of G- parabolic Higgs bundles

We shall deal also with non-reductive Lie groups!

Applications

L. Muñoz, Hodge polynomials of the SL(2,C)-character variety of an elliptic curve with two marked points, Internat. J. Math. 25 (2014), no. 14, 1450–125.

$$e(\mathfrak{M}^1_{C_1,C_2}(\mathrm{SL}_2(\mathbb{C}))) = q^4 + 2q^3 + 6q^2 + 2q + 1$$

Boden-Yokogawa: $P_c(\mathfrak{M}^1_{C_1,C_2}(\mathrm{SL}_2(\mathbb{C})))(t) = t^8 + 3t^6 + 2t^5 + 10t^4$

Thm (Biquard-Jardim): The moduli space of doubly periodic instantons is isomorphic algebraically to the moduli space of non-strongly parabolic Higgs bundles with two punctures on a genus 1 curve.

Non-strongly parabolic… no known Morse theory

Strongly parabolic $|\lambda| = 1$ non-strongly parabolic $|\lambda| \neq 1$ but all character varieties are diffeomorphic

Applications

	10	2	3	0	1	
						1
						2
						6
						2
						1

Applications

$$e(\mathfrak{M}_I^1(\mathsf{SL}_2(\mathbb{C}))) = q^2 + 1$$

$$\mathscr{M}_{Higgs}(\mathsf{SL}_2(\mathbb{C})) \cong \Sigma_g \times \mathbb{C}/\mathbb{Z}_2 \quad e(\mathscr{M}_{Higgs}(\mathsf{SL}_2(\mathbb{C}))) = q^2 + q$$

$$G = \mathsf{SL}_2(\mathbb{C}) \quad {}^LG = \mathsf{PGL}_2(\mathbb{C})$$

(L., Muñoz, Newstead) Mirror symmetry does not hold for character varieties with non-semisimple holonomy $J^{\pm}$

J. Martinez, e-polynomials of $\mathsf{PGL}_2(\mathbb{C})$-character varieties of surface groups, arxiv:1705.04649

M. Mauri, Topological mirror symmetry for rank two character varieties of surface groups, arxiv:2101.04659

Happy birthday Peter!