E-Polynomials of the $\mathrm{SL}_2(\mathbb{C})$-character varieties of surface groups

(joint works with A. González-Prieto, M. Logares, J. Martínez, P. Newstead)

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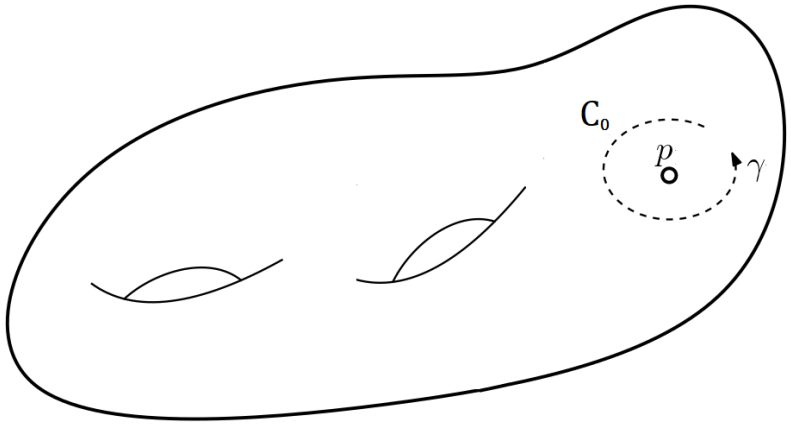
1. Character varieties of surfaces

$G = \mathrm{SL}_r(\mathbb{C}), \mathrm{GL}_r(\mathbb{C}), \mathrm{PGL}_r(\mathbb{C})$
Σ_g compact oriented surface of genus $g \geq 1$
$p \in \Sigma_g$, γ loop around p
$C_0 \in G$, $\mathcal{C} = [C_0]$ conjugacy class.



Character variety (moduli space of representations):

$$\begin{aligned}
\mathfrak{M}^g_{C_0}(G) &= \{\rho : \pi_1(\Sigma_g - \{p\}) \to G \,|\, \rho(\gamma) \in \mathcal{C}\}//G \\
&= \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} | \prod [A_i, B_i] \in \mathcal{C}\}//G \\
&= \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} | \prod [A_i, B_i] = C_0\}//\mathrm{Stab}(C_0)
\end{aligned}$$

Character varieties of surfaces

Fact

$\mathfrak{M}^g_{e^{2\pi i\, d/r}\mathrm{Id}}(G)$ parametrizes local systems over Σ_g on a degree d, rank r complex vector bundle.

We also deal with the representation varieties:

$$\begin{aligned}
\mathfrak{R}^g_{C_0}(G) &= \{\rho : \pi_1(\Sigma_g - \{p\}) \to G \,|\, \rho(\gamma) \in C\} \\
&= \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \,|\, \prod [A_i, B_i] \in C\} \\
\overline{\mathfrak{R}}^g_{C_0}(G) &= \{\rho : \pi_1(\Sigma_g - \{p\}) \to G \,|\, \rho(\gamma) = C_0\} \\
&= \{(A_1, B_1, \ldots, A_g, B_g) \in G^{2g} \,|\, \prod [A_i, B_i] = C_0\}
\end{aligned}$$

E-polynomial

The E-polynomial (Hodge-Deligne, Deligne-Serre polynomial) is:

$$e(Z) = e(Z)(u,v) := \sum (-1)^k h_c^{k,p,q}(Z) u^p v^q$$

Basic properties

- If $Z = \bigsqcup_{i=1}^n Z_i$, then $e(Z) = \sum_{i=1}^n e(Z_i)$.
- $e(Y \times Z) = e(Y)e(Z)$
- If $F \to X \to Y$ is a fibration in the Zariski topology (or principle bundle) then $e(X) = e(F)e(Y)$
- $e(\mathbb{C}) = q$, where $q = uv$
- $e(\mathrm{SL}_2(\mathbb{C})) = e(\mathrm{PGL}_2(\mathbb{C})) = q^3 - q$
- X of balanced type when $h_c^{k,p,q}(X) = 0$ for $p \neq q$, $e(X)$ depends only on $q := uv$
- If $[X] = P(\mathbb{L}) \in \mathbb{Z}[\mathbb{L}] \subset K(\mathcal{V}\mathrm{ar}_{\mathbb{C}})$, then X is balanced $e(X) = P(q)$

$G = \mathrm{SL}_2(\mathbb{C})$

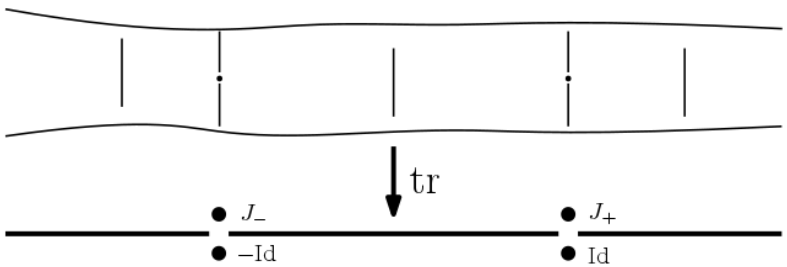
Conjugacy classes

- $\{\mathrm{Id}\}, \{-\mathrm{Id}\}$
- $[J_+], [J_-], J_+ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, J_- = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ Jordan forms

  $\mathrm{Stab}(J_\pm) = U = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix}, a \in \mathbb{C}^*, b \in \mathbb{C} \right\}$, $e(U) = q(q-1)$
- $[C_\lambda], C_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, diagonalizable, $\lambda \in \mathrm{B} = \mathbb{C}^* - \{\pm 1\}$

  $\mathrm{Stab}(C_\lambda) = D = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in \mathbb{C}^* \right\}$, $e(D) = (q-1)^2$

Parametrized by $\mathrm{B}/\mathbb{Z}_2$, $\lambda \sim \lambda^{-1}$, $t = \lambda + \lambda^{-1} \in \mathbb{C} - \{\pm 2\}$



Objective

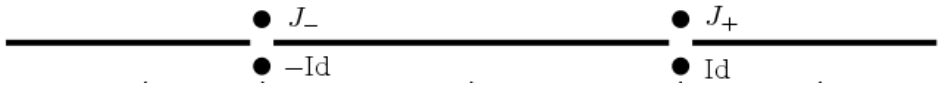
Goal

For $G = \mathrm{SL}_r(\mathbb{C}), \mathrm{GL}_r(\mathbb{C}), \mathrm{PGL}_r(\mathbb{C})$, $C_d = e^{2\pi i\, d/r}\mathrm{Id}$
Compute the E-polynomials of character varieties $\mathfrak{M}^g_{C_d}(G)$

Specific goal

For $G = \mathrm{SL}_2(\mathbb{C})$, any conjugacy class C_0
Compute the E-polynomials of character varieties $\mathfrak{M}^g_{C_0}(\mathrm{SL}_2(\mathbb{C}))$

- For diagonalizable C_λ, get "parabolic" representations
- Role of Jordan forms $J_\pm$
- Monodromy around ± 2



Arithmetic method (Hausel, Rodríguez-Villegas, Letellier, Mereb, ...)

Compute $e(\mathfrak{M}_C^g(\mathrm{SL}_2(\mathbb{C})))$ by:

- Weil conjectures
- Use Katz's result on point counting: Let X be a $\mathbb{Z}$-scheme. Suppose that there exists a polynomial $P(x) \in \mathbb{Z}[x]$ such that $|X(\mathbb{F}_q)| = P(q)$, $q = p^n$. Then $e(X) = P(q)$, $q = uv$.
- Answer written in terms of the character tables of $\mathrm{GL}_2(\mathbb{F}_q), \mathrm{SL}_2(\mathbb{F}_q)$ (combinatorial data of partitions of n)

Geometric method (Logares, Martínez, Muñoz, Newstead)

Compute $e(\mathfrak{M}_C^g(\mathrm{SL}_2(\mathbb{C})))$ by:

- Explicit description of matrices
- Stratifying varieties of representations
- Behaviour of E-polynomials for fibrations (with monodromy)
- Inductive process on $g \geq 1$

Equivariant E-polynomial

F finite group acting on a variety X
$R(F) =$ representation ring of F
$S_1 = T, S_2, \ldots, S_\ell$ irreducible F-representations,
$\ell =$ number of conjugacy classes of F
Then $H^*(X) \in R(F)$

Definition (Equivariant E-polynomial)

$$e_F(X) = \sum (-1)^k H^{k,p,q}(X) u^p v^q \in R(F)[u,v]$$

$e(X) = \dim e_F(X)$
$e(X/F) = \langle e_F(X), T \rangle$

Example: $F = \mathbb{Z}_2$

$e_{\mathbb{Z}_2}(X) = e(X)^+ T + e(X)^- N$
$e((X \times Y)/\mathbb{Z}_2) = e(X)^+ e(Y)^+ + e(X)^- e(Y)^-$

Hodge monodromy representation

Fibration locally trivial in analytic topology $F \longrightarrow Z \xrightarrow{\pi} B$
Suppose F of balanced type.

Monodromy representation

$$\rho : \pi_1(B) \longrightarrow \mathrm{GL}(H_c^{k,p,p}(F)).$$

Definition (Hodge monodromy representation)

Suppose $\Gamma = \mathrm{im}\,(\rho)$ is a finite group

$$R(Z) := \sum (-1)^k H_c^{k,p,p}(F)\, q^p \in R(\Gamma)[q]\,.$$

Hodge monodromy representation

Finite covering $B_\rho \to B$ associated to Γ (with trivial monodromy)

Proposition

Suppose that B_ρ is of balanced type. Then Z is of balanced type. There is a $\mathbb{Z}[q]$-linear map $E : R(\Gamma)[q] \to \mathbb{Z}[q]$ satisfying $E(R(Z)) = e(Z)$.

Write $s_i(q) = E(S_i)$, $S_i \in R(F)$, $1 \leq i \leq \ell$. For

$$R(Z) = a_1(q)S_1 + \ldots + a_\ell(q)S_\ell,$$

then

$$e(Z) = a_1(q)s_1(q) + \ldots + a_\ell(q)s_\ell(q).$$

Example

If $B = \mathbb{C} - \{k \text{ points}\}$, then $e(Z) = (q-1)\, e(F)^{inv} - (k-1)e(F)$

2. Stratifying representation varieties

Let $g \geq 1$ be any natural number. We define the following sets:

- $\overline{X}_0^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod [A_i, B_i] = \mathrm{Id}\}$
- $\overline{X}_1^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod [A_i, B_i] = -\mathrm{Id}\}$
- $\overline{X}_2^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod [A_i, B_i] = J_+\}$
  $X_2^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod [A_i, B_i] \sim J_+\} \cong \overline{X}_2^g \times \mathrm{PGL}_2(\mathbb{C})/U$
- $\overline{X}_3^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod [A_i, B_i] = J_-\}$
  $X_3^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod [A_i, B_i] \sim J_-\} \cong \overline{X}_3^g \times \mathrm{PGL}_2(\mathbb{C})/U$
- $\overline{X}_{4,\lambda}^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod [A_i, B_i] = C_\lambda\}$
  $X_{4,\lambda}^g = \{(A_1, B_1, \ldots, A_g, B_g) \mid \prod [A_i, B_i] \sim C_\lambda\} \cong \overline{X}_{4,\lambda}^g \times \mathrm{PGL}_2(\mathbb{C})/D$
- $\overline{X}_4^g = \{(A_1, B_1, \ldots, A_g, B_g, \lambda) \mid \prod [A_i, B_i] = C_\lambda\}$

Fibration: $\overline{X}_{4,\lambda}^g \to \overline{X}_4^g \to \mathrm{B} = \mathbb{C}^* - \{\pm 1\}$

Encoding E-polynomial information for genus g

- $e_0^g = e(\overline{X}_0^g)$
- $e_1^g = e(\overline{X}_1^g)$
- $e_2^g = e(\overline{X}_2^g)$
- $e_3^g = e(\overline{X}_3^g)$

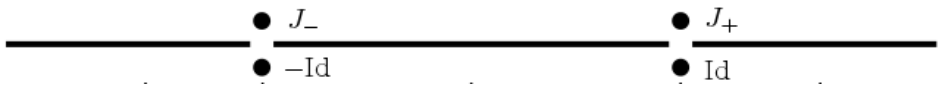
We have fibrations:

$$\overline{X}_4^g \longrightarrow \mathrm{B} = \mathbb{C}^* - \{\pm 1\}$$

$$\overline{X}_4^g/\mathbb{Z}_2 \longrightarrow \mathrm{B}/\mathbb{Z}_2 \cong \mathbb{C} - \{\pm 2\}$$

Monodromy group: $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, generated by loops $\gamma_{\pm 2}$ around ± 2
$R(\Gamma)$ generated by T, $S_{\pm 2}$, $S_0 = S_2 \otimes S_{-2}$

$$R(\overline{X}_4^g/\mathbb{Z}_2) = a_g T + b_g S_2 + c_g S_{-2} + d_g S_0$$



Genus g vector

$v_g = (e_0^g, e_1^g, e_2^g, e_3^g, a_g, b_g, c_g, d_g)$

E-polynomial of the fibration

$\overline{X}_4 \to \mathrm{B} = \mathbb{C}^* - \{\pm 1\}$
$\overline{X}_4/\mathbb{Z}_2 \to \mathrm{B}/\mathbb{Z}_2 = \mathbb{C} - \{\pm 2\}$

$$\begin{array}{ccccc} \mathbb{C}^* & \longrightarrow & \mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_4 & \longrightarrow & \widetilde{X}_4 \\ \| & & \downarrow & & \downarrow 2{:}1 \\ \mathbb{C}^* & \longrightarrow & (\mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_4)/\mathbb{Z}_2 & \longrightarrow & X_4 \end{array}$$

$$\begin{aligned} e((\mathrm{PGL}(2,\mathbb{C}) \times \overline{X}_4)/\mathbb{Z}_2) &= e(\widetilde{X}_4)^+ e(\mathbb{C}^*)^+ + e(\widetilde{X}_4)^- e(\mathbb{C}^*)^- \\ &= e(X_4)q - (e(\widetilde{X}_4) - e(X_4)) \\ &= (q+1)e(X_4) - q(q+1)e(\overline{X}_4) \end{aligned}$$

$$\begin{aligned} &= e(\mathrm{PGL}(2,\mathbb{C}))^+ e(\overline{X}_4)^+ + e(\mathrm{PGL}(2,\mathbb{C}))^- e(\overline{X}_4)^- \\ &= q(q^2-1)e(\overline{X}_4/\mathbb{Z}_2)\,. \end{aligned}$$

$\Rightarrow$ from $R(\overline{X}_4/\mathbb{Z}_2)$ we get $e(X_4)$.

Starting case. Genus $g = 1$

$\overline{X}^1_0 = \{(A, B) \mid [A, B] = \mathrm{Id}\}$
$\overline{X}^1_1 = \{(A, B) \mid [A, B] = -\mathrm{Id}\}$
$\overline{X}^1_2 = \{(A, B) \mid [A, B] = J_+\}$
$\overline{X}^1_3 = \{(A, B) \mid [A, B] = J_-\}$
$\overline{X}^1_{4,\lambda} = \{(A, B) \mid [A, B] = C_\lambda\}$

- $e^1_0 = e(\overline{X}^1_0) = q^4 + 4q^3 - q^2 - 4q$
- $e^1_1 = e(\overline{X}^1_1) = q^3 - q$
- $e^1_2 = e(\overline{X}^1_2) = q^3 - 2q^2 - 3q \qquad (e(\mathfrak{M}^1_{J_+}) = q^2 - 3q - 3)$
- $e^1_3 = e(\overline{X}^1_3) = q^3 + 3q^2 \qquad (e(\mathfrak{M}^1_{J_-}) = q^2 + 3q)$
- $R(\overline{X}^1_4/\mathbb{Z}_2) = q^3 T - 3qS_2 + 3q^2 S_{-2} - S_0$
  $a_1 = q^3$, $b_1 = -3q$, $c_1 = 3q^2$, $d_1 = -1$

Hence $e(\overline{X}^1_{4,\lambda}) = q^3 - 3q + 3q^2 - 1 \qquad (e(\mathfrak{M}^1_{C_\lambda}) = q^2 + 4q + 1)$

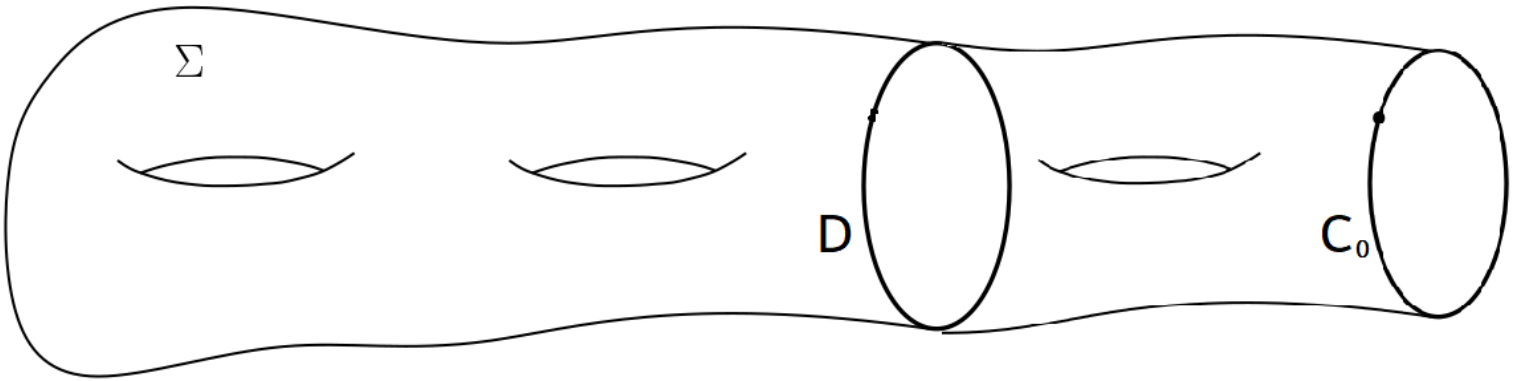
Also get $e(\overline{X}^1_4), e(\overline{X}^1_4/\mathbb{Z}_2), e(X^1_4), \ldots$

Main tool

Connected sum decomposition $\Sigma_{g+1} = \Sigma_g \# \Sigma_1$

Gluing formula for the E-polynomials with monodromies $D = \mathrm{Id}, -\mathrm{Id}, J_+, J_-$ or C_λ around the neck

$$\prod_{i=1}^{g+1} [A_i, B_i] = C_0 \iff \prod_{i=1}^{g} [A_i, B_i] = D = C_0 [B_{g+1}, A_{g+1}]$$



Recursion. Example e_0^{g+1}

To compute $e(\overline{X}_0^{g+1})$, we have the contributions:

- $\{\prod_{i=1}^g [A_i, B_i] = [B_{g+1}, A_{g+1}] = \mathrm{Id}\} \cong \overline{X}_0^g \times \overline{X}_0^1$
- $\{\prod_{i=1}^g [A_i, B_i] = [B_{g+1}, A_{g+1}] = -\mathrm{Id}\} \cong \overline{X}_1^g \times \overline{X}_1^1$
- $\{\prod_{i=1}^g [A_i, B_i] = [B_{g+1}, A_{g+1}] \sim J_+\} \cong \overline{X}_2^g \times \overline{X}_2^1 \times \mathrm{PGL}_2(\mathbb{C})/U$
- $\{\prod_{i=1}^g [A_i, B_i] = [B_{g+1}, A_{g+1}] \sim J_-\} \cong \overline{X}_3^g \times \overline{X}_3^1 \times \mathrm{PGL}_2(\mathbb{C})/U$
- $\overline{W}_4 = \{(\prod_{i=1}^g [A_i, B_i] = [B_{g+1}, A_{g+1}] = C_\lambda, \lambda)\}$

This has Hodge monodromy representation:

$$\begin{aligned} R(\overline{W}_4/\mathbb{Z}_2) &= R(\overline{X}_4^g/\mathbb{Z}_2) \otimes R(\overline{X}_4^1/\mathbb{Z}_2) \\ &= (a_g T + b_g S_2 + c_g S_{-2} + d_g S_0) \otimes \\ &\quad \otimes (q^3 T - 3q S_2 + 3q^2 S_{-2} - S_0) \end{aligned}$$

$$\begin{aligned} e_0^{g+1} = {} & (q^4 + 4q^3 - q^2 - 4q) e_g^0 + (q^3 - q) e_g^1 + (q^3 - 2q^2 - 3q) e_g^2 \\ & + (q^3 + 3q^2) e_g^3 + q^3 a_g - 3q\, b_g + 3q^2 c_g - d_g \end{aligned}$$

Recursion matrix: $v_{g+1} = M \cdot v_g$

$M =$

$$\begin{pmatrix}
q^4+4q^3 & q^3-q & q^5-2q^4-4q^3 & q^5+3q^4 & q^6-2q^5-4q^4 & -q^5-4q^4 & 2q^5-7q^4-3q^3 & -5q^4-q^3 \\
-q^2-4q & & +2q^2+3q & -q^3-3q^2 & +3q^2+2q & +4q^2+q & +7q^2+q & +5q^2+q \\
q^3-q & q^4+4q^3 & q^5+3q^4 & q^5-2q^4-4q^3 & q^6-2q^5-4q^4 & 2q^5-7q^4-3q^3 & -q^5-4q^4 & -5q^4-q^3 \\
 & -q^2-4q & -q^3-3q^2 & +2q^2+3q & +3q^2+2q & +7q^2+q & +4q^2+q & +5q^2+q \\
q^3-2q^2 & q^3+3q^2 & q^5+q^4 & q^5-3q^3 & q^6-2q^5-3q^4 & -q^5+2q^4 & -q^5-q^4 & -2q^4-q^3 \\
-3q & & +3q^2+3q & -6q^2 & +q^3+3q^2 & -4q^3+3q^2 & -4q^3+6q^2 & +3q^2 \\
q^3+3q^2 & q^3-2q^2 & q^5-3q^3 & q^5+q^4 & q^6-2q^5-3q^4 & -q^5-q^4 & -q^5+2q^4 & -2q^4-q^3 \\
 & -3q & -6q^2 & +3q^2+3q & +q^3+3q^2 & -4q^3+6q^2 & -4q^3+3q^2 & +3q^2 \\
q^3 & q^3 & q^5-3q^3 & q^5-3q^3 & q^6-2q^5-2q^4 & -q^5-q^4 & -q^5-q^4 & -2q^4 \\
 & & & & +4q^3+q^2 & +2q^3 & +2q^3 & \\
-3q & 3q^2 & 3q^2+3 & -6q^2 & -3q^3+3q^2 & 4q^4-6q^3+4q^2 & -8q^3+6q^2 & -3q^3+3q^2 \\
3q^2 & -3q & -6q^2 & 3q^3+3q & -3q^3+3q^2 & -8q^3+6q^2 & 4q^4-6q^3+4q^2 & -3q^3+3q^2 \\
-1 & -1 & 2q^2 & 2q^2 & -4q^2+2 & -2q^2+q+1 & -2q^2+q+1 & q^4-2q^2 \\
 & & & & & & & +2q+1
\end{pmatrix}$$

We solve as $v_g = M^g \cdot v_0$, with $v_0 = (1,0,0,0,0,0,0,0)$.

Theorem (Martínez-Muñoz)

The E-polynomials for $\mathfrak{M}_C^g(\mathrm{SL}_2(\mathbb{C}))$ *are:*

$$\begin{aligned}
e(\mathfrak{M}^g_{\mathrm{Id}}) &= (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - q(q^2-q)^{2g-2} - 2^{2g}q^{2g-2} \\
&\quad + \frac{1}{2}q^{2g-2}(q+2^{2g}-1)((q+1)^{2g-2} + (q-1)^{2g-2}) \\
&\quad + \frac{1}{2}q((q+1)^{2g-1} + (q-1)^{2g-1}) \\
e(\mathfrak{M}^g_{-\mathrm{Id}}) &= (q^3-q)^{2g-2} + (q^2-1)^{2g-2} - 2^{2g-1}(q^2+q)^{2g-2} \\
&\quad + (2^{2g-1}-1)(q^2-q)^{2g-2} \\
e(\mathfrak{M}^g_{J_+}) &= (q^3-q)^{2g-2}(q^2-1) + \frac{1}{2}q^{2g-2}(q-1)((q-1)^{2g-1} - (q+1)^{2g-1}) \\
&\quad + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} - 2^{2g-1}(q+1)(q^2+q)^{2g-2} \\
e(\mathfrak{M}^g_{J_-}) &= (q^3-q)^{2g-2}(q^2-1) + (2^{2g-1}-1)(q-1)(q^2-q)^{2g-2} \\
&\quad + 2^{2g-1}(q+1)(q^2+q)^{2g-2} \\
e(\mathfrak{M}^g_{C_\lambda}) &= (q^3-q)^{2g-2}(q^2+q) + (q^2-1)^{2g-2}(q+1) \\
&\quad + (2^{2g}-2)(q^2-q)^{2g-2}q
\end{aligned}$$

4. Topological Quantum Field Theories

Paradigm

The previous recursion hints to the existence of a TQFT

Let $n \geq 1$. The category $\mathbf{Bord}_n$ is:

- **Objects**: $(n-1)$-dimensional closed manifolds (maybe empty).
- **Morphisms**: $W : M_1 \to M_2$ is a n-dimensional compact manifold W such that $\partial W = M_1 \sqcup M_2$, up to boundary preserving diffeomorphism (bordism).

Topological Quantum Field Theory

A TQFT is a **lax monoidal** symmetric functor

$$Z : \mathbf{Bord}_n \to R\text{-}\mathbf{Mod}.$$

Lax monoidality: $\Delta_{M_1,M_2} : Z(M_1) \otimes_R Z(M_2) \longrightarrow Z(M_1 \sqcup M_2)$

Algebraic topology via TQFTs

Goal

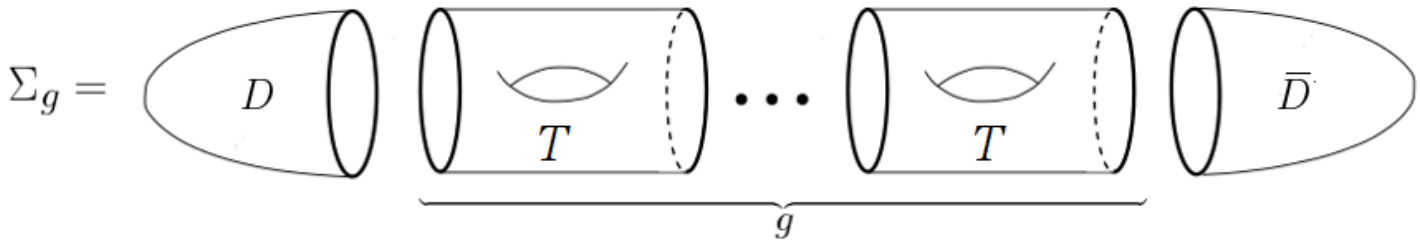
For a closed manifold W, we want to compute an algebraic invariant $\chi(W) \in R$.

Key idea: Quantize the invariant

Construct a TQFT

$$Z : \mathbf{Bord}_n \to R\text{-}\mathbf{Mod}$$

such that $Z(\emptyset) = R$ and for $W : \emptyset \to \emptyset$ closed, $Z(W)(1) = \chi(W)$.



$$\chi(\Sigma_g) = Z(\Sigma_g)(1) = Z(\overline{D}) \circ Z(T)^g \circ Z(D)(1)$$

TQFT for representation varieties

Goal

Construct a TQFT that computes $e(\mathfrak{R}_{\Sigma_g}(G))$

Remark

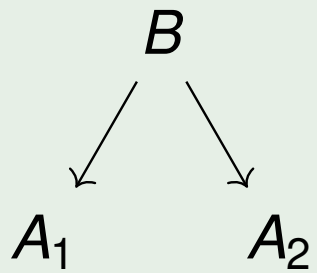
Instead of $\mathbf{Bord}_n$, we will use $\mathbf{Bordp}_n$, the category of n-dimensional bordisms with a marked finite subset.

$$Z : \mathbf{Bordp}_n \xrightarrow[\mathcal{F}_{\mathcal{G}}]{\text{Field theory}} \mathrm{Span}(\mathbf{Var}_{\mathbb{C}}) \xrightarrow[\mathcal{Q}_{\mathcal{A}}]{\text{Quantization}} R\text{-}\mathbf{Mod}$$

Category: $\mathrm{Span}(\mathcal{C})$

Objects: $A \in \mathrm{Obj}(\mathcal{C})$

Morphisms:



$\mathbf{Bord}_n$ is a subcategory of $\mathrm{Span}(\mathbf{Diff}^{\mathrm{op}})$.

5a. Field theory for representation varieties

$(M, A) \in \mathbf{Bordp}_n$
M a compact manifold, $A \subset M$ finite set
$\Pi(M, A) =$ fundamental groupoid

Field theory functor $\mathcal{F}_{\mathcal{G}}$

We define $\mathcal{G}(M, A) = \mathrm{Hom}(\Pi(M, A), G) \in \mathbf{Var}_{\mathbb{C}}$
For a cobordism $(W, A) : (M_1, A_1) \to (M_2, A_2)$,

we have a span

$$\begin{array}{ccc} & \mathcal{G}(W, A) & \\ \swarrow & & \searrow \\ \mathcal{G}(M_1, A_1) & & \mathcal{G}(M_2, A_2) \end{array}$$

5b. Quantization via mixed Hodge modules

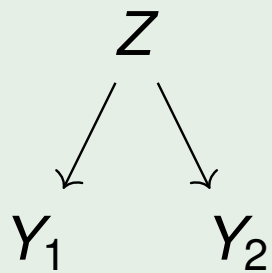
Saito's mixed Hodge modules

$$\text{Complex algebraic variety } X \implies \text{Abelian monoidal category } \mathcal{M}_X \implies \text{Ring } \mathrm{K}\mathcal{M}_X$$

- $\mathcal{M}_X$ contains variations of Hodge structures on X
- $\mathcal{M}_\star = \mathbf{MHS}$, category of mixed Hodge structures
- Hodge monodromy representation can be understood in this context
- For the projection $c_X : X \to \star$, we have $(c_X)_!(1) = [H_c^\bullet(X, \mathbb{Q})]$

Quantization functor $\mathcal{Q}_{\mathrm{K}\mathcal{M}}$

For a span



we take $(i_2)_! \circ (i_1)^* : \mathrm{K}\mathcal{M}_{Y_1} \to \mathrm{K}\mathcal{M}_{Y_2}$

6. Quantization method

TQFT for representation varieties

$Z = \mathcal{Q}_{\mathrm{K}\mathcal{M}} \circ \mathcal{F}_{\mathcal{G}} : \mathbf{Bordp}_n \to \mathrm{K}\mathbf{MHS}\text{-}\mathbf{Mod}$

Let W be a closed manifold and $A \subset W$ finite
Bordism: $W : \emptyset \to \emptyset$

$Z(W, A) : \mathrm{K}\mathbf{MHS} \to \mathrm{K}\mathbf{MHS}$

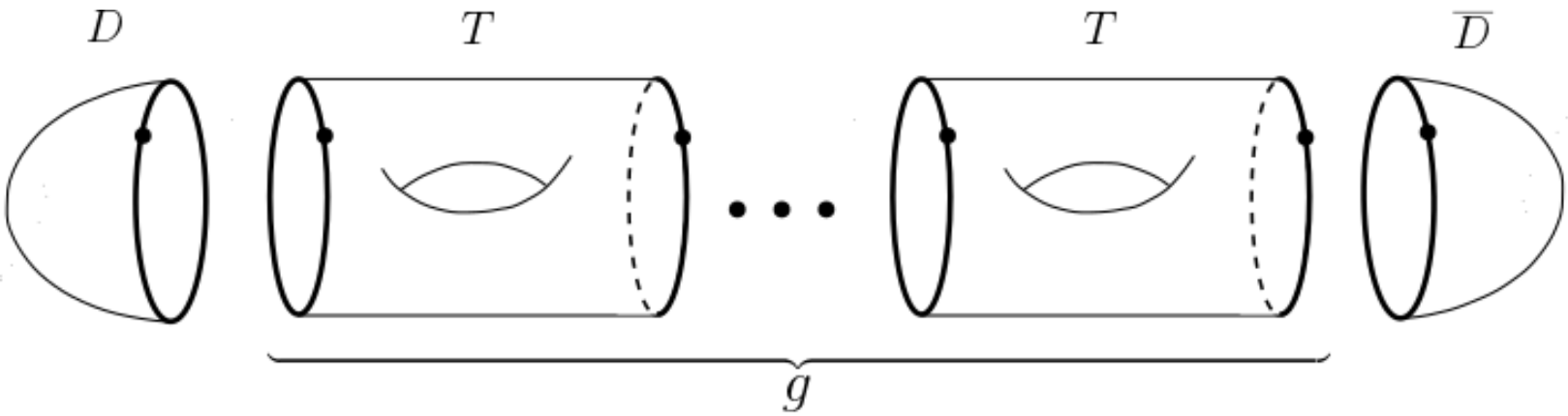
$Z(W, A)(1) = [H_c^\bullet(\mathfrak{R}_W(G), \mathbb{Q})] \times [H_c^\bullet(G, \mathbb{Q})]^{|A|-1}$
$e(Z(W, A)(1)) = e(G)^{|A|-1} e(\mathfrak{R}_W(G))$

Theorem (González-Prieto, Logares, Muñoz)

There exists a lax monoidal TQFT computing E-polynomials of representation varieties.

TQFT for surfaces

The genus g closed orientable surface Σ_g with $g+1$ marked points is written as $\Sigma_g = \overline{D} \circ T^g \circ D$



Hence

$$e(\mathfrak{R}_{\Sigma_g}(G)) = \frac{1}{e(G)^g}\left[Z(\overline{D}) \circ Z(T)^g \circ Z(D)(1)\right]$$

Explicit computations

$Z = \mathcal{Q}_{\mathrm{K}\mathcal{M}} \circ \mathcal{F}_{\mathcal{G}}$

Field Theory morphisms

$$\mathcal{F}_{\mathcal{G}}(D) = \left[1 \longleftarrow 1 \xrightarrow{i} G\right] \qquad \mathcal{F}_{\mathcal{G}}(\overline{D}) = \left[G \xleftarrow{i} 1 \longrightarrow 1\right]$$

$$\mathcal{F}_{\mathcal{G}}(T) = \begin{bmatrix} G & \xleftarrow{p} & G^4 & \xrightarrow{s} & G \\ g & \longleftarrow & (g, g_1, g_2, h) & \longmapsto & hg[g_1, g_2]h^{-1} \end{bmatrix}$$

Recall $\mathrm{K}\mathcal{M}_{S^1} = \mathrm{K}\mathcal{M}_G$, so $Z(T) : \mathrm{K}\mathcal{M}_G \to \mathrm{K}\mathcal{M}_G$

Morphisms of the TQFT

$$Z(D) = i_! \qquad Z(\overline{D}) = i^* \qquad Z(T) = s_! p^*$$

For $G = \mathrm{SL}_2(\mathbb{C})$, we have $\mathrm{C} = \mathbb{Q}[q]^8 \subset \mathrm{K}\mathcal{M}_G$, called "core"
Then $Z(D)(1) = v_0 \in V$, and $Z(T) = M : \mathrm{C} \to \mathrm{C}$

7. Affine group

$$\mathrm{AGL}_1(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix} \middle| a \in \mathbb{C}, b \in \mathbb{C}^* \right\}$$

Theorem (González-Prieto, Logares, Muñoz)

Let $G = \mathrm{AGL}_1(\mathbb{C})$ and $g \geq 1$
Then $e(\mathfrak{R}_{\Sigma_g}(\mathrm{AGL}_1(\mathbb{C})) = q^{2g-1}(q-1)^{2g} + q^{2g} - q^{2g-1}$

$\mathrm{C} = \mathbb{Q}[q]^2 \subset \mathrm{K}\mathcal{M}_{S^1}$, generated by

$$i_!\mathbb{Q}_\star \qquad i : \{\mathrm{Id}\} \hookrightarrow \mathrm{AGL}_1(\mathbb{C}),$$
$$j_!\mathbb{Q}_{\mathrm{ASO}_1^*(\mathbb{C})} \qquad j : \mathrm{ASO}_1^*(\mathbb{C}) = \mathrm{ASO}_1(\mathbb{C}) - \{\mathrm{Id}\} \hookrightarrow \mathrm{AGL}_1(\mathbb{C})$$

$Z(D)(1) = (1,0)$
$Z(\overline{D})(a,b) = a$
$$Z(L) = q(q-1)\begin{pmatrix} q^3 - q^2 & q^4 - 3q^3 + 2q^2 \\ q^3 - 2q^2 & q^4 - 3q^3 + 3q^2 \end{pmatrix}$$

Affine group

Observation

Knowing the form of $Z(L)$ (TQFT theoretical result),
can compute $Z(L)$ by testing on primes and using arithmetic method!

$$e(\mathfrak{R}_{\Sigma_g}(\mathrm{AGL}_1(\mathbb{C})) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q^3 - q^2 & q^4 - 3q^3 + 2q^2 \\ q^3 - 2q^2 & q^4 - 3q^3 + 3q^2 \end{pmatrix}^g \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q-1 & q-1 \\ -1 & q-1 \end{pmatrix} \begin{pmatrix} q^{2g} & 0 \\ 0 & q^{2g}(q-1)^{2g} \end{pmatrix} \begin{pmatrix} q-1 & q-1 \\ -1 & q-1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= q^{2g-1}\left((q-1)^{2g} + q - 1\right)$$

Problem

$\mathrm{AGL}_1(\mathbb{C})$ is not reductive
$\mathfrak{M}_{\Sigma_g}(\mathrm{AGL}_1(\mathbb{C})) = \mathfrak{R}_{\Sigma_g}(\mathrm{AGL}_1(\mathbb{C}))//\mathrm{AGL}_1(\mathbb{C}) = (\mathbb{C}^*)^{2g}$
It loses much information

Affine group

Theorem (González-Prieto, Hablicksek, Vogel)

There exists a lax monoidal TQFT

$$Z : \mathbf{Bordp}_n \to \mathrm{K}\left(\mathbf{Stack}/BG\right)\text{-}\mathbf{Mod},$$

computing virtual classes of G-character stacks

$$Z(W)(1) = [\mathfrak{M}_W(G) \to BG] \in \mathrm{K}\left(\mathbf{Stack}/BG\right)$$

$$Z(L) = \begin{pmatrix} 1 + q\,(q-2)\,[\mathbb{G}_a/G] + (q+1)\,[\mathbb{G}_m/G] & q\,(q-2)\,[\mathrm{AGL}_1(\mathbb{C})/G] \\ q\,(q-2)\,[\mathbb{G}_a/G] & q^2 + q\,(q-1)\,(q-2)\,[\mathbb{G}_a/G] \end{pmatrix}$$

The virtual class of the $\mathrm{AGL}_1(\mathbb{C})$-character stack is

$$[\mathfrak{M}_{\Sigma_g}(\mathrm{AGL}_1(\mathbb{C}))] = BG + ((q-1)^{2g} - 1)[\mathbb{G}_a/G] + \frac{q^{2g}-1}{q-1}[\mathbb{G}_m/G] + \frac{(q^{2g-2}-1)\left((q-1)^{2g}-1\right)}{q-1}[\mathrm{AGL}_1(\mathbb{C})/G].$$

8. Other results

Theorem (Martínez)

E-polynomials (via recursion) of $\mathrm{PGL}_2(\mathbb{C})$-character varieties of surface groups.

Theorem (Hablicksek, Vogel)

E-polynomials (via TQFT) of upper triangular matrices up to rank 4.

Theorem (Vogel)

E-polynomials (via TQFT) of $\mathrm{SL}_2(\mathbb{C})$-character varieties of non-orientable surfaces.

Method highly promising ...

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