THE MODULI STACK OF PRINCIPAL ρ-SHEAVES AND GIESEKER HARDER-NARASIMHAN FILTRATIONS

Alfonso Zamora [1]

Universidad Politécnica de Madrid, Spain

VBAC Moduli Spaces and Vector Bundles - New Trends
On the occasion of Peter Newstead's 80^{th} birthday
25-29 July 2022, University of Warwick, UK

[1] joint work with A. Fernández-Herrero (Cornell, USA) and T. L. Gómez (ICMAT, Spain)

Outline

1. Preliminaries and Motivation
2. Stratification on the stack of principal ρ-sheaves
3. Gieseker Harder-Narasimhan filtrations
4. Example of Gieseker Harder-Narasimhan filtration

Outline

1 PRELIMINARIES AND MOTIVATION

2 STRATIFICATION ON THE STACK OF PRINCIPAL ρ-SHEAVES

3 GIESEKER HARDER-NARASIMHAN FILTRATIONS

4 EXAMPLE OF GIESEKER HARDER-NARASIMHAN FILTRATION

Torsion-free sheaves and Hilbert polynomials

- k field of characteristic zero (many results also true over char $k > 0$).
- $(X, \mathcal{O}(1))$ smooth projective geometrically connected variety over k of $\dim X = d$ with a very ample line bundle.

For $\dim X = d \geq 1$ and a torsion-free sheaf E on X, let $E(n) = E \otimes \mathcal{O}(n)$. Let $\deg E$ be its degree and $\mathrm{rk}\, E$ its rank.

- **Hilbert polynomial**:

$$P_E(n) = \chi(E(n)) = \sum_{i=0}^{d} (-1)^i h^i(E(n)) = a_d n^d + a_{d-1} n^{d-1} + \cdots + a_2 n^2 + a_1 n + a_0$$

where $a_d \propto \mathrm{rk}\, E$ and $a_{d-1} \propto \deg E$.

- **Reduced Hilbert polynomial**:

$$\overline{p}_E(n) = \frac{P_E(n)}{\mathrm{rk}(E)} \propto n^d + \mu_{d-1} n^{d-1} + \cdots + \mu_2 n^2 + \mu_1 n + \mu_0$$

μ_i are **intermediate i^{th} slopes** and $\mu_{d-1} \propto \mu(E) = \frac{\deg E}{\mathrm{rk}\, E}$, usual slope.

Slope and Gieseker stability

Definition

A torsion-free sheaf E over X is **slope** resp. **Gieseker (semi)stable** if for every $0 \neq F \subsetneq E$, it is

$$\mu(F) \underset{(\leq)}{<} \mu(E) \quad \text{resp.} \quad \overline{p}_F(n) \underset{(\leq)}{<} \overline{p}_E(n),$$

and **slope** resp. **Gieseker unstable** otherwise.

slope stable $\Rightarrow$ ***Gieseker stable*** $\Rightarrow$ ***Gieseker semistable*** $\Rightarrow$ ***slope semistable***

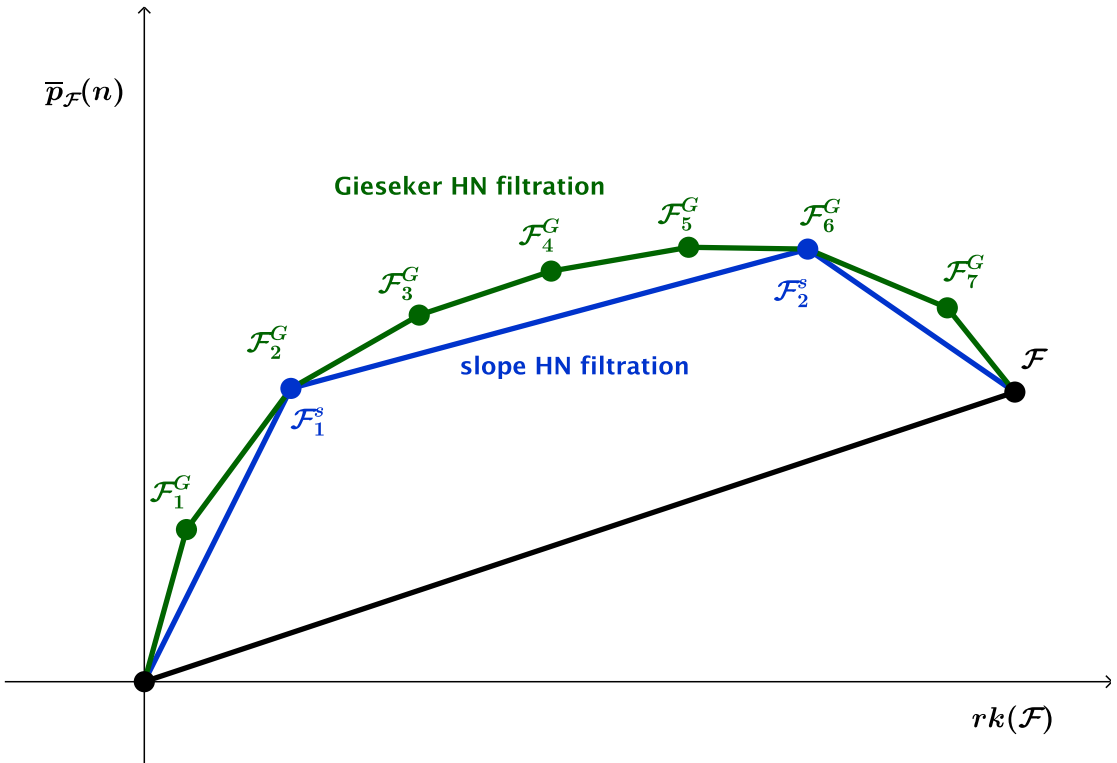
Gieseker stability is a refinement of slope stability.

Slope and Gieseker Harder-Narasimhan filtration

THEOREM

*A torsion-free sheaf E over X has a unique **slope** resp. **Gieseker** **Harder-Narasimhan filtration** $0 \subset E_1 \subset E_2 \subset \cdots \subset E_t = E$ such that:*

- $\mu(E_i/E_{i-1})$ *resp.* $\overline{p}_{E_i/E_{i-1}}(n)$ *are a decreasing sequence.*
- *All quotients E_i/E_{i-1} are **slope** resp. **Gieseker** semistable.*



***Gieseker HN** refines **slope HN** filtration.*

Moduli of principal G-bundles over curves

- G linear reductive algebraic group over k.
- V k-vector space of $\dim r$.
- Fix $\rho : G \longrightarrow \mathrm{GL}(V)$ faithful representation.

For $\dim X = 1$, [Ramanathan'76] constructs a **projective moduli space of semistable principal G-bundles using Geometric Invariant Theory (GIT)**, with the adjoint representation

$$\rho = \mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g})$$

and a **stability condition of slope type**:

*A principal G-bundle $\mathcal{G}$ over $(X, \mathcal{O}(1))$ is **(semi)stable** if for every reduction of structure group $\mathcal{G}_P$ to a proper parabolic subgroup $P \subset G$ and all dominant characters χ of P, the associated line bundle $\chi_*(\mathcal{G}_P)$ satisfies*
$\deg \chi_*(\mathcal{G}_P) \underset{(\leq)}{<} 0.$

Moduli of principal G-bundles over higher dimensional varieties

For $\dim X > 1$, [Gómez-Langer-Schmitt-Sols'02'05'08] consider degenerations of principal G-bundles to **principal ρ-sheaves** $(\mathcal{P}, \mathcal{E}, \psi)$:

Given $\rho : G \to \mathrm{GL}(V)$, $\mathcal{P}$ is principal G-bundle on big open $U \subset X$ ($\mathrm{codim}\, X \backslash U \geq 2$), $\mathcal{E}$ is torsion free sheaf on X and isomorphism $\psi : \mathcal{P}(\rho) \simeq \mathcal{E}|_U$.

and construct projective **moduli spaces using GIT** with a **stability condition of Hilbert polynomial or Gieseker type**:

*$(\mathcal{P}, \mathcal{E}, \psi)$ is **(semi)stable** if all filtrations $\mathcal{E}_\bullet \subset \mathcal{E}$ Killing orthogonal (i.e. $\mathcal{E}_i^\perp = \mathcal{E}_{-i-1}$ and $[\mathcal{E}_i, \mathcal{E}_j] \subset \mathcal{E}_{i+j}^{\vee\vee}$), it is $\sum (P_{\mathcal{E}_i} \mathrm{rk}\, \mathcal{E} - P_{\mathcal{E}} \mathrm{rk}\, \mathcal{E}_i) \underset{(\leq)}{<} 0$.*

*Goal 1: Give a **stack-theoretic interpretation of the moduli space of principal bundles** avoiding the intrincations of GIT.*

Harder-Narasimhan for G-bundles

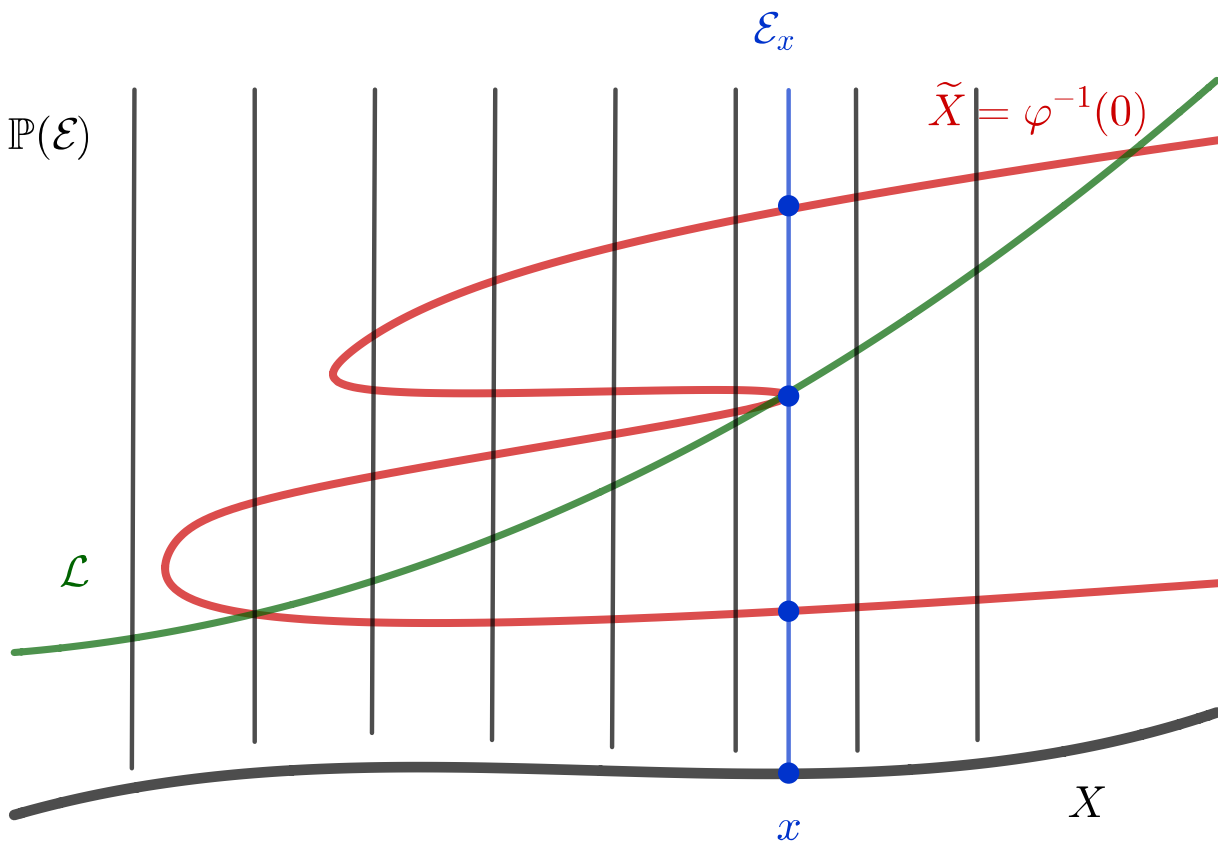
- [Ramanathan'76, Behrend'95, Biswas-Holla'04] construct an analog for principal G-bundles over a curve X: **canonical reduction to a parabolic** $P \subset G$ where the associated principal L-bundle is semistable, $P = L \ltimes U$.
- For $\dim X > 1$, [Anchouche-Azad-Biswas'02] show the existence of **canonical reductions for slope stability conditions**.
  - [Gurjar-Nitsure'16'18] relative canonical reductions and **stratification of the stack by Harder-Narasimhan types**.

*Goal 2: Give a **new notion of Gieseker or polynomial Harder-Narasimhan reduction for principal ρ-sheaves**.*

Previous attempts: Harder-Narasimhan for rank 2 tensors

THEOREM (Z.'15)

Let $\mathcal{E}$ be a $\mathrm{rk}\,2$ *torsion-free sheaf and M a line bundle over X. Every* $\mathrm{rk}\,2$ ***tensor*** *$(\mathcal{E}, \varphi : \mathcal{E}^{\otimes s} \to M)$ has a unique* ***Harder-Narasimhan*** *subtensor $(\mathcal{L}, \varphi|_{\mathcal{L}}) \subset (\mathcal{E}, \varphi)$ maximizing certain polynomial function. If φ is symmetric and X is a curve, the stability condition can be rephrased in terms of intersection theory on ruled surfaces.*



Previous attempts: relate Gieseker HN for principal bundles with HN for underlying bundle

The Harder-Narasimhan filtration of the underlying vector bundle and the canonical reduction of the principal bundle coincide for $d = 1$; but with a Gieseker stability condition, they do differ for $d > 1$.

THEOREM (BISWAS-Z.'16)

Example of an $\mathrm{O}(6, \mathbb{C})$-bundle $(V, \varphi : V \otimes V \to \mathcal{O}_X)$ on a K3 surface X such that the Gieseker Harder-Narasimhan filtration of V does not correspond to any parabolic reduction of the principal $\mathrm{O}(6, \mathbb{C})$-bundle, therefore it cannot relate to the canonical reduction of (V, φ).

New approach: understand Gieseker stability globally in the stack

- [Gómez-Sols-Z.'15, Hoskins-Kirwan'12] **GIT gives natural stratifications on a truncation of the stack, but not globally.** Harder-Narasimhan filtrations come asymptotically as Hesselink-Kempf-Ness filtrations in a sequence of GIT problems (Quot schemes change with twist).
- [Halpern-Leistner'14] **Theory of Θ-stability and Θ-stratifications** by locally closed substacks (analog to the Hesselink-Kempf-Ness stratification of GIT quotients). Define canonical HN filtrations to each unstable point **in the whole stack**.
  - Active area of research with applications to different moduli problems: bundles, varieties.

Outline

1 PRELIMINARIES AND MOTIVATION

2 STRATIFICATION ON THE STACK OF PRINCIPAL ρ-SHEAVES

3 GIESEKER HARDER-NARASIMHAN FILTRATIONS

4 EXAMPLE OF GIESEKER HARDER-NARASIMHAN FILTRATION

Stack of principal ρ-sheaves

Definition

Given a k-scheme S, a **principal ρ-sheaf** on $X_S := X \times_k S$ is $(\mathcal{F}, \sigma)$ where

- $\mathcal{F}$ is a torsion-free sheaf of rank r over X_S, locally free over $U \subset X$ big ($\operatorname{codim} X \backslash U \geq 2$) open.
- σ is a G-reduction of structure group of the $\mathrm{GL}(V)$-bundle $\mathcal{F}|_U$.

Define $\mathrm{Bun}_\rho(X)$, *pseudofunctor from affine schemes into groupoids:*

$$\mathrm{Bun}_\rho(X)(S) = [\textit{groupoid of } \rho\textit{-sheaves } (\mathcal{F}, \sigma) \textit{ on } X_S]_{/\mathcal{F}_1 \simeq \mathcal{F}_2 \textit{ identifying } \sigma_1 = \sigma_2}$$

Analog of 1-parameter subgroups: Θ-filtrations on stacks

*Want to **generalize the Hilbert-Mumford criterion** to stacks to define stability and Harder-Narasimhan filtrations.*

- Let $\mathcal{M}$ be a stack over k and $K \supset k$ a field extension.
- Define $\Theta_K = [\mathbb{A}^1_K/\mathbb{G}_m]$, action of $\mathbb{G}_m$ with weight -1 on variable t.

Origin $0 \in \mathbb{A}^1_K$ ***unique closed point***; 1 *(image of the* K*-point* $1 \in \mathbb{A}^1_K$*) is an **open point**.*

Definition

Let $p \in \mathcal{M}$ be a K-point. A **Θ-filtration** of the point p is a morphism of stacks $f : \Theta_K \longrightarrow \mathcal{M}$ with an isomorphism $f(1) \simeq p$.

- Graded point $f|_{[0/\mathbb{G}_m]}$ is called the **associated graded** of the filtration.

Rees construction turns Θ-filtrations into $\mathbb{Z}$-filtrations

*If $\mathcal{M} = \mathrm{Coh}^{tf}(X)$ (stack of torsion-free coherent sheaves on X) a Θ-filtration of $\mathcal{F}$ is equivalent (by **Rees construction**) to a $\mathbb{Z}$-filtration $(\mathcal{F}_m)_{m\in\mathbb{Z}}$:*
$\cdots = 0 = 0 = \mathcal{F}_l \subset \mathcal{F}_{l+1} \subset \cdots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \mathcal{F} = \mathcal{F} = \cdots$
with $\mathcal{F}_m/\mathcal{F}_{m-1}$ torsion-free and associated graded $\bigoplus_{m\in\mathbb{Z}} \mathcal{F}_m/\mathcal{F}_{m-1}$.

For the stack $\mathrm{Bun}_\rho(X)$, a **Θ-filtration** $f : \Theta_K \to \mathrm{Bun}_\rho(X)$ of $(\mathcal{F}, \sigma)$ is $((\mathcal{F}_m)_{m\in\mathbb{Z}}, \mathcal{P})$:

- $(\mathcal{F}_m)_{m\in\mathbb{Z}}$ is a Θ-filtration of the torsion-free sheaf $\mathcal{F}$.
- $U \hookrightarrow X_K$ is a maximal big open set where the associated graded $\bigoplus_{m\in\mathbb{Z}} \mathcal{F}_m/\mathcal{F}_{m-1}$ is locally free.
- $\mathcal{P}$ is a G-bundle on $U \times_K \Theta_K$ such that $\rho_*(\mathcal{P})$ (associated $\mathrm{GL}(V)$-bundle) equals the restriction $(\mathcal{F}_m|_U)_{m\in\mathbb{Z}}$.

$$\mathrm{Filt}(\mathcal{F}, \sigma) = [\textit{groupoid of } \Theta\textit{-filtrations of } (\mathcal{F}, \sigma) \leftrightarrow ((\mathcal{F}_m)_{m\in\mathbb{Z}}, \mathcal{P})]_{/\sim}$$

Polynomial numerical invariant on $\mathrm{Bun}_\rho(X)$ and stability

Definition (polynomial numerical invariant)

For each filtration $f : \Theta_K \to \mathrm{Bun}_\rho(X) \in \mathrm{Filt}(\mathcal{F}, \sigma)$ of $(\mathcal{F}, \sigma)$, define

$$\nu(f) = \frac{\text{weight of certain line bundle}}{\text{quadratic norm on graded points}} = \frac{\text{weight } (L_n|_0)}{\sqrt{b\left(f|_{[0/\mathbb{G}_m]}\right)}}$$

Using [Halpern-Leistner, Herrero, Jones'21], relate with $\mathrm{Coh}_r^{tf}(X)$:

$$\nu(f) = \frac{\sum_{m\in\mathbb{Z}} m \cdot (\overline{p}_{\mathcal{F}_m/\mathcal{F}_{m-1}} - \overline{p}_{\mathcal{F}}) \cdot \mathrm{rk}(\mathcal{F}_m/\mathcal{F}_{m-1})}{\sqrt{\sum_{m\in\mathbb{Z}} m^2 \,\mathrm{rk}(\mathcal{F}_m/\mathcal{F}_{m-1})}}.$$

This defines a **notion of stability**: a ρ-sheaf $(\mathcal{F}, \sigma)$ is Gieseker semistable if for every filtration $f \in \mathrm{Filt}(\mathcal{F}, \sigma)$ the function $\nu(f)$ is non-positive:

$$\sum_{m\in\mathbb{Z}} m \cdot (\overline{p}_{\mathcal{F}_m/\mathcal{F}_{m-1}} - \overline{p}_{\mathcal{F}}) \cdot \mathrm{rk}(\mathcal{F}_m/\mathcal{F}_{m-1}) \leq 0$$

and Gieseker unstable otherwise (generalizes **Hilbert-Mumford criterion**).

This definition coincides with [Gómez-Langer-Schmitt-Sols'02'05'08].

Θ-stratification and good moduli spaces

THEOREM (HALPERN-LEISTNER'14)

Suppose that the numerical invariant ν on a stack $\mathcal{M}$ satisfies:

1. ***Θ-monotonicity*** *(codim 2 over DVR extension property)*
2. ***S-monotonicity*** *(separatedness, S-equivalence)*
3. ***HN-boundedness*** *($\forall$ destabilizing filtrations $\nu(f) > 0$ can find a more destabilizing one $\nu(f') > \nu(f) > 0$ in a fixed bounded family).*
4. ***Boundedness*** *of the semistable locus.*
5. *Existence part of valuation criterion for* ***properness****.*

Then ν induces a ***Θ-stratification*** *on the stack $\mathcal{M}$ (1 and 3) and the semistable locus admits a* ***separated*** *(2)* ***good moduli space*** *(in the sense of [Alper'13]) (4) that is* ***proper*** *(5).*

Main result 1: Stack construction of moduli of principal ρ-sheaves

THEOREM (GÓMEZ, FERNÁNDEZ-HERRERO, Z.'21)

Let ν be the polynomial numerical invariant defined above on $\mathrm{Bun}_\rho(X)$.

1. *ν is Θ-monotone and satisfies the HN-boundedness condition (hence defines a **Θ-stratification on** $\mathrm{Bun}_\rho(X)$), plus ν is S-monotone.*
2. *The substack $\mathrm{Bun}_\rho(X)^{ss}$ of Gieseker semistable ρ-sheaves satisfies the existence part of the valuative criterion for properness and, fixing the degree ϑ and the Hilbert polynomial P, $\mathrm{Bun}_\rho(X)^{ss,P}_{\vartheta}$ is bounded.*
3. *Therefore the stack $\mathrm{Bun}_\rho(X)$ admits a **separated good** (in the sense of Alper) **moduli space** which is **proper**.*

*Provides an **alternative stack intrinsic approach** to the constructions of [Gómez-Langer-Schmitt-Sols'02'05'08].*

*Can **extend setting** to faithful representations $\rho : G \longrightarrow \prod_{i=1}^{b} \mathrm{GL}(V^i)$ to allow reductive groups G with large center.*

Outline

1 PRELIMINARIES AND MOTIVATION

2 STRATIFICATION ON THE STACK OF PRINCIPAL ρ-SHEAVES

3 GIESEKER HARDER-NARASIMHAN FILTRATIONS

4 EXAMPLE OF GIESEKER HARDER-NARASIMHAN FILTRATION

Θ-stratification and leading term HN filtration

Given a Gieseker unstable ρ-sheaf $(\mathcal{F},\sigma) \in \mathrm{Bun}_\rho(X)$, $\exists$ Θ-filtration $f \in \mathrm{Filt}(\mathcal{F},\sigma)$ such that $\nu(f) > 0$. Define

$$A^\nu(\mathcal{F},\sigma) := \sup_{f \in \mathrm{Filt}(\mathcal{F},\sigma)} \nu(f)$$

- The **Θ-stratification** defined by ν is $\mathrm{Bun}_\rho(X) = \bigsqcup_{A\in\mathbb{Q}[n]} \mathcal{M}_A$ where

$$\mathcal{M}_A = \{(\mathcal{F},\sigma) \in \mathrm{Bun}_\rho(X) : A^\nu(\mathcal{F},\sigma) = A\}, \quad \text{with } \mathcal{M}_0 = \mathrm{Bun}_\rho(X)^{ss}.$$

Proposition (Halpern-Leistner'14, Gómez, Fernández-Herrero, Z.'21)

*Every $(\mathcal{F},\sigma) \in \mathcal{M}_A$ has a unique canonical Θ-filtration f_{can} (equivalently, a pair $((\mathcal{F}_m)_{m\in\mathbb{Z}}, \mathcal{P})$) such that $\nu(f_{can}) = A$, called the **leading term Harder-Narasimhan filtration**.*

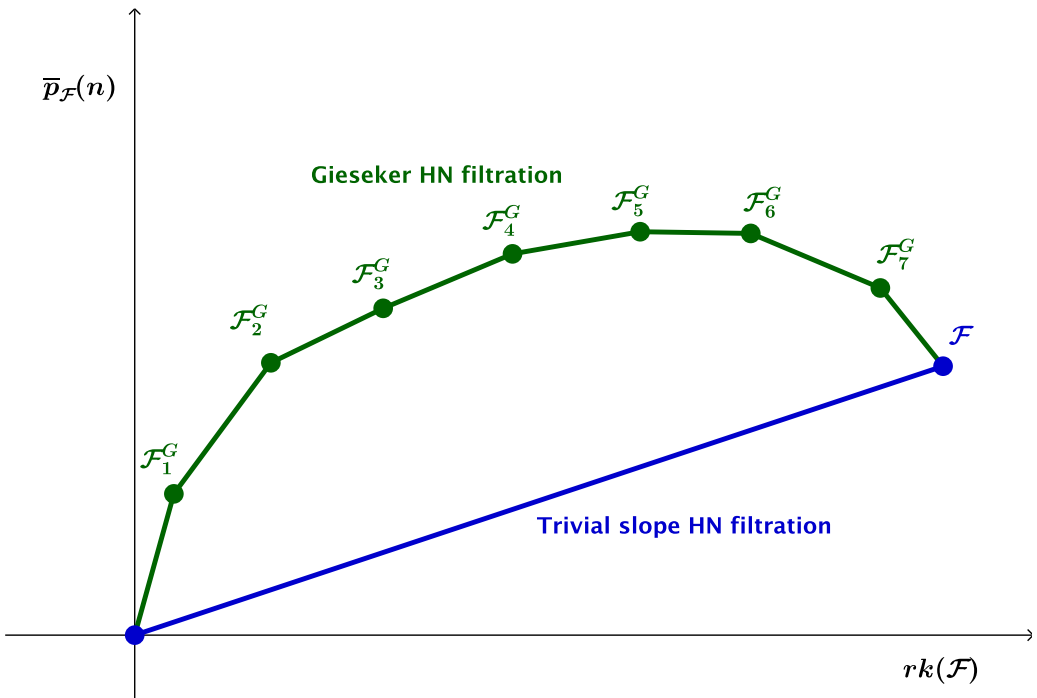
Leading term HN filtration for torsion-free $\mathcal{F}$

Idea of Leading term HN filtration for torsion-free sheaves $\mathcal{F}$: case $G = \mathrm{GL}(V)$ and $\rho = \mathrm{id}$.

- Suppose $\mathcal{F}$ is slope semistable, but Gieseker unstable, then:

Gieseker HN filtration: $0 \subset \mathcal{F}_1^G \subset \mathcal{F}_2^G \subset \cdots \subset \mathcal{F}_n^G = \mathcal{F}$

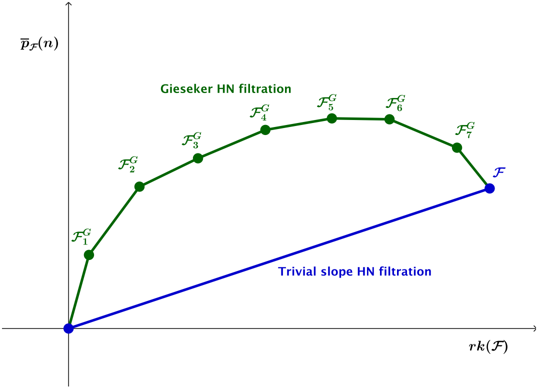
slope HN filtration: $0 \subset \mathcal{F}$



$$\begin{array}{ccccccccc} 0 \subset & \mathcal{F}_1^G & \subset & \mathcal{F}_2^G & \subset & \mathcal{F}_3^G & \subset \cdots \subset & \mathcal{F}_{n-1}^G \subset \mathcal{F} \\ & \overline{p}_{\mathcal{F}_1^G} & > & \overline{p}_{\mathcal{F}_2^G/\mathcal{F}_1^G} & > & \overline{p}_{\mathcal{F}_3^G/\mathcal{F}_2^G} & > \cdots > & \overline{p}_{\mathcal{F}_n^G/\mathcal{F}_{n-1}^G} \end{array}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\begin{array}{ccccccccc} 0 \subset & & & & & & & \mathcal{F} \\ & \mu_{d-1}(\mathcal{F}_1^G) & = & \cdots & = & \mu_{d-1}(\mathcal{F}_i^G/\mathcal{F}_{i-1}^G) & = \cdots = & \mu_{d-1}(\mathcal{F}_n^G/\mathcal{F}_{n-1}^G) \end{array}$$



In the reduced Hilbert polynomial of $\mathcal{F}$:

$$\overline{p}_{\mathcal{F}}(n) = n^d + \mu_{d-1}(\mathcal{F})n^{d-1} + \mu_{d-2}(\mathcal{F})n^{d-2} + \mu_{d-3}(\mathcal{F})n^{d-3} + \cdots$$

all inequalities come from i^{th} slopes μ_i with $i < d-1$.

Leading term HN filtration comes from maximizing $\nu(f)$

Recall the polynomial numerical invariant

$$\nu(f) = \frac{\sum_{m\in\mathbb{Z}} m \cdot (\bar{p}_{\mathcal{F}_m/\mathcal{F}_{m-1}} - \bar{p}_{\mathcal{F}}) \cdot \mathrm{rk}(\mathcal{F}_m/\mathcal{F}_{m-1})}{\sqrt{\sum_{m\in\mathbb{Z}} m^2 \,\mathrm{rk}(\mathcal{F}_m/\mathcal{F}_{m-1})}}$$

and let f be the unique canonical Θ-filtration of $\mathcal{F}$ maximizing ν.

- Degree of numerator of $\nu(f)$ is highest index i for which there are differences (at least one) for i^{th} slopes $\mu_i(\mathcal{F}_m/\mathcal{F}_{m-1})$ of the quotients.
- Maximum of ν achieved for the (unique) filtration with largest possible i (**leading index**) such that the **leading slopes**:

$$\mu_i(\mathcal{F}_1^G) \geq \mu_i(\mathcal{F}_2^G/\mathcal{F}_1^G) \geq \cdots \geq \mu_i(\mathcal{F}_n^G/\mathcal{F}_{n-1}^G)$$

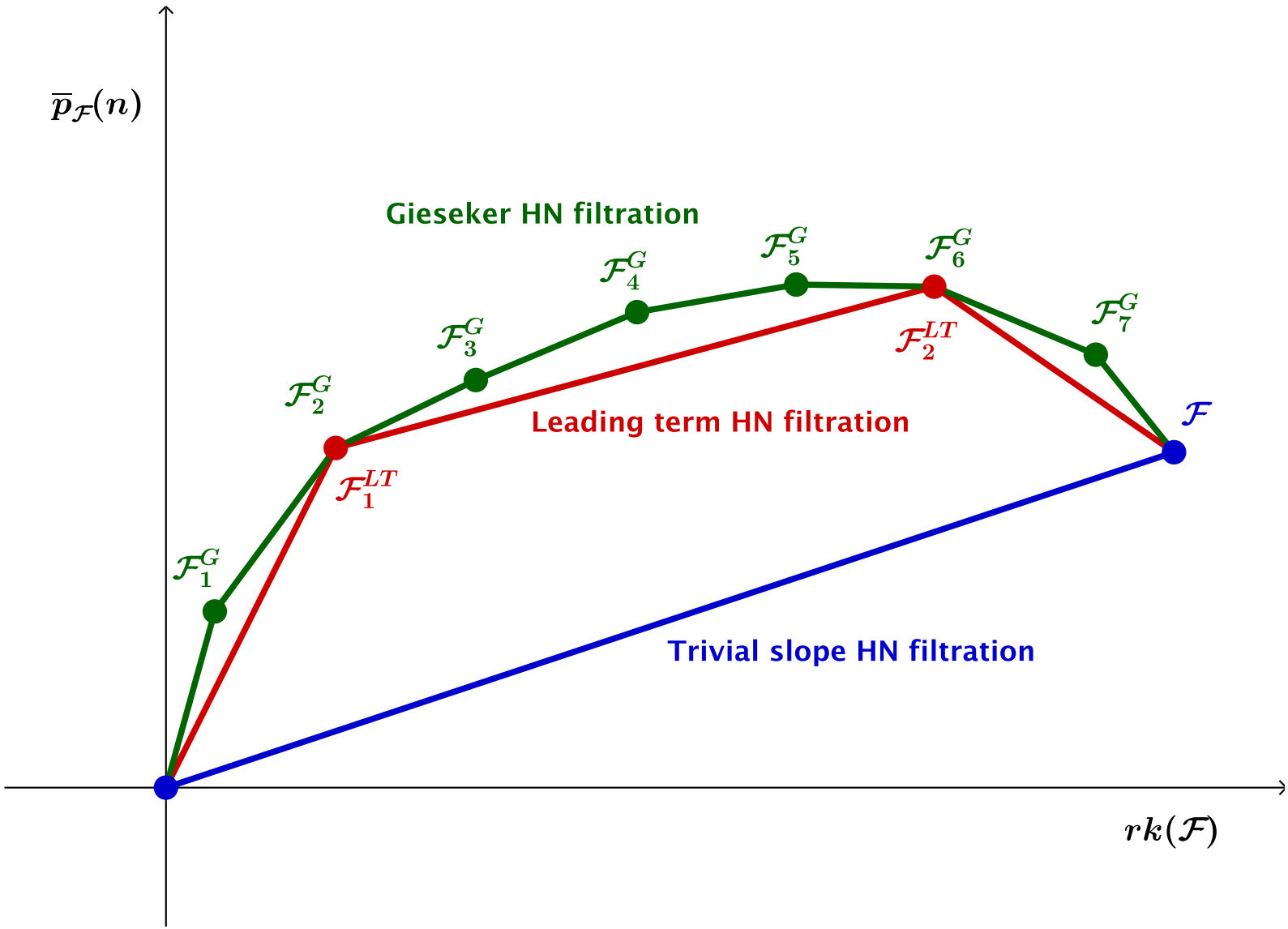
with, at least, one strict inequality.

Leading term Harder-Narasimhan filtration

Leading term Harder-Narasimhan filtration *of* $\mathcal{F}$ *is* $0 \subset \mathcal{F}_1^{LT} \subset \mathcal{F}_2^{LT} \subset \cdots \subset \mathcal{F}$ *where* $\mathcal{F}_k^{LT} = \mathcal{F}_{h_k}^G$ *come from jumps* $\mu_i(\mathcal{F}_{h_k}^G/\mathcal{F}_{h_{k-1}}^G) > \mu_i(\mathcal{F}_{h_{k+1}}^G/\mathcal{F}_{h_k}^G)$.

$$
\begin{array}{ccccccccc}
0 \subset & \mathcal{F}_1^G & \subset & \mathcal{F}_2^G & \subset & \mathcal{F}_3^G & \subset \cdots \subset & \mathcal{F}_{n-1}^G & \subset \mathcal{F} \\
 & \overline{p}_{\mathcal{F}_1^G} & > & \overline{p}_{\mathcal{F}_2^G/\mathcal{F}_1^G} & > & \overline{p}_{\mathcal{F}_3^G/\mathcal{F}_2^G} & > \cdots > & \overline{p}_{\mathcal{F}_n^G/\mathcal{F}_{n-1}^G} & \\
\| & & & & & & & \| & \\
0 \subset & \mathcal{F}_1^{\mathrm{LT}} & & \subset & & \mathcal{F}_2^{\mathrm{LT}} & \subset \cdots \subset & \mathcal{F} & \\
 & \mu_i(\mathcal{F}_1^{\mathrm{LT}}) & & > & & \mu_i(\mathcal{F}_2^{\mathrm{LT}}/\mathcal{F}_1^{\mathrm{LT}}) & > \cdots > & \mu_i(\mathcal{F}_j^{\mathrm{LT}}/\mathcal{F}_{j-1}^{\mathrm{LT}}) & \\
\| & & & & & & & \| & \\
0 \subset & & & & & & & \mathcal{F} & \\
 & \mu_{d-1}(\mathcal{F}_1^G) & = & \cdots & = & \mu_{d-1}(\mathcal{F}_i^G/\mathcal{F}_{i-1}^G) & = \cdots = & \mu_{d-1}(\mathcal{F}_n^G/\mathcal{F}_{n-1}^G) &
\end{array}
$$

Shatz polygon of the Leading Term HN filtration



Split case: Θ-filtrations come from parabolic reductions

Recall canonical Θ-filtration f_{can} is equivalent to a pair $((\mathcal{F}_m)_{m\in\mathbb{Z}}, \mathcal{P})$.
If G_K is split, following [Heinloth'17] reduction $\mathcal{P}$ (which is G-bundle on $U \times_K \Theta_K$) comes from a **weighted parabolic reduction** on a maximal big open set:

- *$\mathcal{G}$ is the restriction of $\mathcal{P}$ to $1 \in \Theta_K$.*
- *cocharacter $\lambda : \mathbb{G}_m \longrightarrow G$ defines parabolic subgroup $P_\lambda \subset G$.*
- *$\mathcal{G}_\lambda$ is a reduction of $\mathcal{G}$ to P_λ.*

Applying to the P_λ-bundle $\mathcal{G}_\lambda$ the extension

$$P_\lambda = L_\lambda \ltimes U_\lambda \twoheadrightarrow L_\lambda \hookrightarrow G_K$$

*matches the **associated graded** $f|_{[0/\mathbb{G}_m]} : [\mathrm{Spec}(K)/\mathbb{G}_m] \to \mathrm{Bun}_\rho(X)$.*

Main result 2: Gieseker-Harder-Narasimhan filtration of a ρ-sheaf $(\mathcal{F}, \sigma)$

THEOREM (GÓMEZ, FERNÁNDEZ-HERRERO, Z.'21)

Every principal ρ-sheaf $(\mathcal{F}, \sigma)$ admits a unique filtration $(\mathcal{F}_m)_{m\in\mathbb{Z}}$ called the ***Gieseker Harder-Narasimhan filtration****. In the split case, the associated Levi sheaf is Gieseker semistable.*

Proof by recursion:

- Obtain 1^{st} leading term HN filtration $f_{can,1} = (\mathcal{F}_m)_{m\in\mathbb{Z}}$ of $(\mathcal{F}, \sigma)$ (in the split case, induced by weighted parabolic reduction $(\lambda_1, \mathcal{G}_{\lambda_1})$).
- Yields $\rho_{\lambda_1} : L_{\lambda_1} \longrightarrow \mathrm{GL}(V_{\lambda_1}) \subset \mathrm{GL}(V)$ (into Levi subgroup) with $f_{can,1}|_0 = (\oplus_{m\in\mathbb{Z}}\mathcal{F}_m/\mathcal{F}_{m-1}, gr(\sigma)) =: gr_1(\mathcal{F}, \sigma)$ 1^{st} Levi ρ_{λ_1}-sheaf.
- If $gr_1(\mathcal{F}, \sigma)$ is not Gieseker semistable repeat the process to get 2^{nd} leading term HN-filtration $f_{can,2}$ and 2^{nd} Levi $gr_2(\mathcal{F}, \sigma)$.
- After q steps, arrive to $gr_q(\mathcal{F}, \sigma)$ q^{th} Levi $\rho_{(\lambda_1,\ldots,\lambda_q)}$-sheaf which is Gieseker semistable.

Gieseker Harder-Narasimhan filtration

- Recursively, Gieseker Harder-Narasimhan filtration is constructed step by step as the leading term HN filtration of a ρ-sheaf, maximizing a polynomial Gieseker stability condition.
- Each step of the algorithm provides the highest index for which intermediate slopes form a decreasing sequence: **first HN condition**.
- In the split case, associated graded Levi is semistable: **second HN condition**.

*For torsion-free sheaves, this **Gieseker Harder-Narasimhan filtration is**, in fact, **the classical one**.*

Outline

1 PRELIMINARIES AND MOTIVATION

2 STRATIFICATION ON THE STACK OF PRINCIPAL ρ-SHEAVES

3 GIESEKER HARDER-NARASIMHAN FILTRATIONS

4 EXAMPLE OF GIESEKER HARDER-NARASIMHAN FILTRATION

Example of Gieseker HN filtration of a principal ρ-sheaf: setting

- Let $G = \mathrm{SO}(7)$ and $\rho : \mathrm{SO}(7) \to \mathrm{GL}(7)$ standard representation.
- Set $X = \mathbb{P}^3$, fix a line $L \subset \mathbb{P}^3$ and two different points $Z = \{p, p'\} \subset \mathbb{P}^3$.
- Let $\mathcal{I}_L$, $\mathcal{I}_p$, and $\mathcal{I}_Z$ ideal sheaves of L, p and Z and set $\mathcal{F} = \mathcal{I}_L \oplus \mathcal{I}_Z \oplus \mathcal{I}_p \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$, $\mathrm{rk}\,\mathcal{F} = 7$.

$\varphi : \mathcal{F} \otimes \mathcal{F} \to \mathcal{O}$ symmetric bilinear defined by

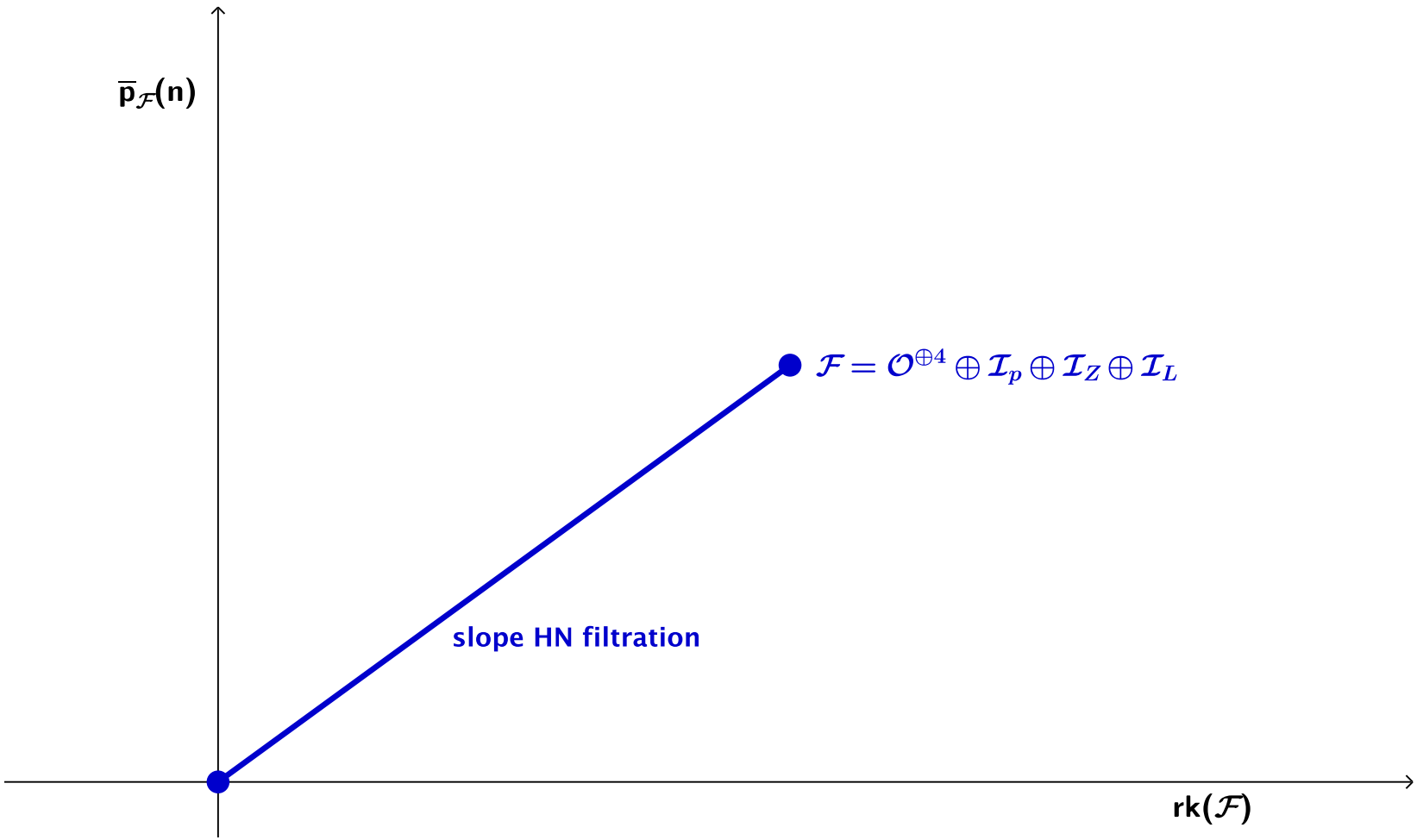
$$\begin{pmatrix} 0 & \cdots & & & & & 1 \\ & & & & & 1 & \\ & & & & 1 & & \\ & & & 1 & & & \\ & & 1 & & & & \\ & 1 & & & & & \\ 1 & & & & & \cdots & 0 \end{pmatrix}$$

- Set $U = \mathbb{P}^3 \setminus (L \cup Z)$, then $\mathcal{F}|_U$ is locally free (in fact trivial).
- φ' canonical trivialization of $\det \mathcal{F}$, then $\det \varphi = \varphi'^2$.
- (φ, φ') induces reduction of the (trivial) $\mathrm{GL}(7)$-bundle $\mathcal{F}|_U$ to a (trivial) $\mathrm{SO}(7)$-bundle $\mathcal{G} \to U$, with the canonical isomorphism $\psi : \rho_*(\mathcal{G}) \longrightarrow \mathcal{F}|_U$.

Yields the ***principal ρ-sheaf*** $(\mathcal{F}, \mathcal{G}, \psi)$.

Example: the principal ρ-sheaf is slope semistable

Principal $\mathrm{SO}(7)$-bundle $\mathcal{G} \to U$ is trivial, hence slope semistable (in the sense of [Ramanathan'76]). Therefore **the Harder-Narasimhan (slope) canonical reduction by [Anchouche, Azad, Biswas'02] is trivial**.



Example: Gieseker stability

Hilbert polynomials of rk 1 sheaves in $\mathcal{F} = \mathcal{I}_L \oplus \mathcal{I}_Z \oplus \mathcal{I}_p \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ are:

$$P_{\mathcal{O}}(n) = \frac{1}{6}(n^3 + 6n^2 + 10n + 3) \quad , \quad P_{\mathcal{I}_p}(n) = \frac{1}{6}(n^3 + 6n^2 + 10n - 3)$$
$$P_{\mathcal{I}_Z}(n) = \frac{1}{6}(n^3 + 6n^2 + 10n - 9) \quad , \quad P_{\mathcal{I}_L}(n) = \frac{1}{6}(n^3 + 6n^2 - 6n - 3)$$

The ρ-sheaf $(\mathcal{F}, \mathcal{G}, \psi)$ is **slope semistable**, by checking leading degree 2 terms of Hilbert polynomials:

$$\mu_{d-1}(\mathcal{H}) = \mu_2(\mathcal{H}) = \mu(\mathcal{H}) = 1 \quad \text{for} \quad \mathcal{H} = \mathcal{O}, \mathcal{I}_p, \mathcal{I}_Z, \mathcal{I}_L$$

*But it is **Gieseker unstable** with **leading index** 1 **and leading slope** μ_1:*

$$\exists\, \mathcal{O} \subsetneq \mathcal{F} \quad \textit{such that} \quad \overline{p}_{\mathcal{O}}(n) = \frac{1}{6}n^3 + n^2 + \frac{5}{3}n + \frac{1}{2} >$$

$$\overline{p}_{\mathcal{F}}(n) = \frac{4 \cdot P_{\mathcal{O}}(n) + P_{\mathcal{I}_P}(n) + P_{\mathcal{I}_Z}(n) + P_{\mathcal{I}_L}(n)}{7} = \frac{1}{6}n^3 + n^2 + \frac{9}{7}n - \frac{1}{14}$$

Example: computation of the 1^{st} leading term HN filtration

Leading Term HN filtration will be the unique filtration maximizing the polynomial numerical invariant ν such that:

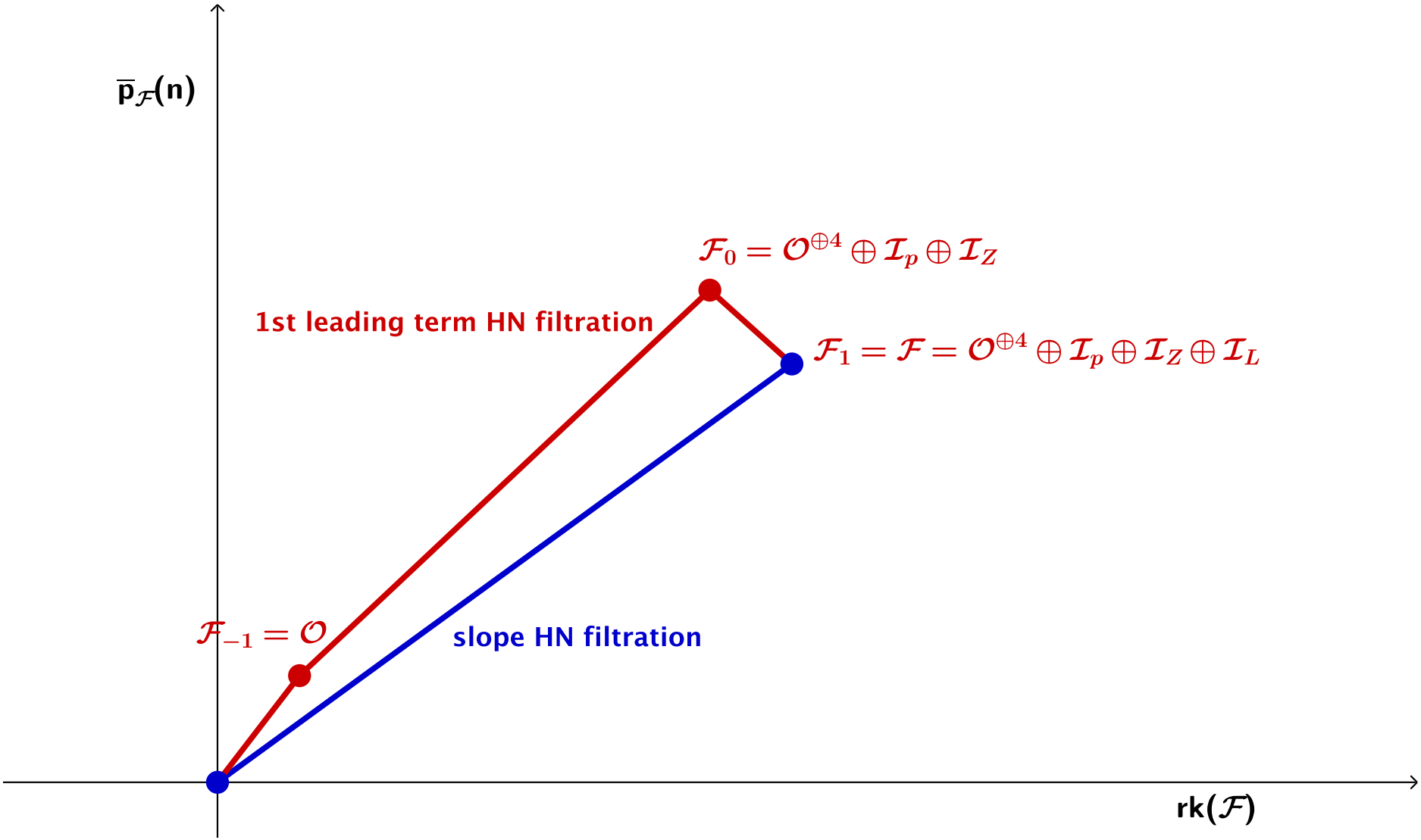
- Comes from cocharacters of $\mathrm{SO}(7)$ of the form $\mathrm{diag}\,(t^a, t^b, t^c, 1, t^{-c}, t^{-b}, t^{-a})$ with $a \geq b \geq c \geq 0$.
- Filtration $(\mathcal{F}_m)_{m\in\mathbb{Z}}$ comes from a reduction to $\mathrm{SO}(7)$): satisfies condition $\mathcal{F}_{-m}^{\perp} = \mathcal{F}_{m-1}$ where $\mathcal{F}_i{}^{\perp} := \ker\left(\mathcal{F} \xrightarrow{\varphi} \mathcal{F}^{\vee} \to \mathcal{F}_i{}^{\vee}\right)$ is the orthogonal of a subsheaf $\mathcal{F}_i \subset \mathcal{F}$.

*The 1^{st} **leading term HN filtration** turns out to be*

$$\overbrace{\mathcal{O}}^{\mathcal{F}_{-1}} \subset \overbrace{\mathcal{I}_Z \oplus \mathcal{I}_p \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}}^{\mathcal{F}_0} \subset \overbrace{\mathcal{F}}^{\mathcal{F}_1}$$

with $\mathcal{F}_{-1}^{\perp} = \mathcal{F}_0$ and cocharacter $\lambda_1 = \mathrm{diag}(t, 1, 1, 1, 1, 1, t^{-1})$.

Shatz polygon of 1^{st} leading term HN filtration



Example: 1^{st} Levi sheaf of the 1^{st} leading term HN filtration

- 1^{st} leading term HN filtration obtained by **cocharacter** $\lambda_1 = \mathrm{diag}(t,1,1,1,1,1,t^{-1})$.
- **Associated 1^{st} Levi** is $L_{\lambda_1} = \mathbb{G}_m \times \mathrm{SO}(5)$ with **representation**

$$\begin{array}{rccc} \rho_{\lambda_1}: & \mathbb{G}_m \times \mathrm{SO}(5) & \longrightarrow & \mathrm{GL}(1)\times\mathrm{GL}(5)\times\mathrm{GL}(1) \subset \mathrm{GL}(7) \\ & (t,g) & \mapsto & (t,g,t^{-1}) \end{array}$$

- **Associated 1^{st} graded ρ_{λ_1}-sheaf for the 1^{st} Levi L_{λ_1}** is

$$\overbrace{\mathcal{O}}^{\mathcal{F}^{-1}:=\mathcal{F}_{-1}} \oplus \overbrace{\mathcal{I}_Z \oplus \mathcal{I}_p \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}}^{\mathcal{F}^{0}:=\mathcal{F}_{0}/\mathcal{F}_{-1}} \oplus \overbrace{\mathcal{I}_L}^{\mathcal{F}^{1}:=\mathcal{F}_{1}/\mathcal{F}_{0}}$$

The $\mathrm{SO}(5)$*-sheaf defined by* $\mathcal{F}^0$ *is* ***Gieseker unstable*** *with* ***leading index*** 0 ***and leading slope*** μ_0*:* $\exists\, \mathcal{O} \subsetneq \mathcal{F}^0$ *such that*

$$\overline{p}_{\mathcal{O}}(n) = \frac{n^3}{6}+n^2+\frac{5}{3}n+\frac{1}{2} > \overline{p}_{\mathcal{F}^0}(n) = \frac{3P_{\mathcal{O}}(n)+P_{\mathcal{I}_P}(n)+P_{\mathcal{I}_Z}(n)}{5} = \frac{n^3}{6}+n^2+\frac{5}{3}n-\frac{3}{5}$$

Then ***compute its*** 2^{nd} ***leading term HN filtration.***

Example: 2^{nd} Levi sheaf of the 2^{nd} leading term HN filtration

*The 2^{nd} **leading term HN filtration** of the $\mathrm{SO}(5)$-sheaf $\mathcal{F}^0$ is*

$$\overbrace{\mathcal{O}}^{\mathcal{F}^0_{-2}} \subset \overbrace{\mathcal{O}\oplus\mathcal{O}}^{\mathcal{F}^0_{-1}} \subset \overbrace{\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}}^{\mathcal{F}^0_{0}} \subset \overbrace{\mathcal{I}_p\oplus\mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}}^{\mathcal{F}^0_{1}} \subset \overbrace{\mathcal{F}^0}^{\mathcal{F}^0_{2}}$$

- Given by the **cocharacter** $\lambda_2 = \mathrm{diag}(t^2, t, 1, t^{-1}, t^{-2})$.
- **Associated 2^{nd} Levi** is $L_{\lambda_2} = \mathbb{G}_m \times (\mathbb{G}_m \times \mathbb{G}_m) \subset \mathbb{G}_m \times \mathrm{SO}(5)$, with representation

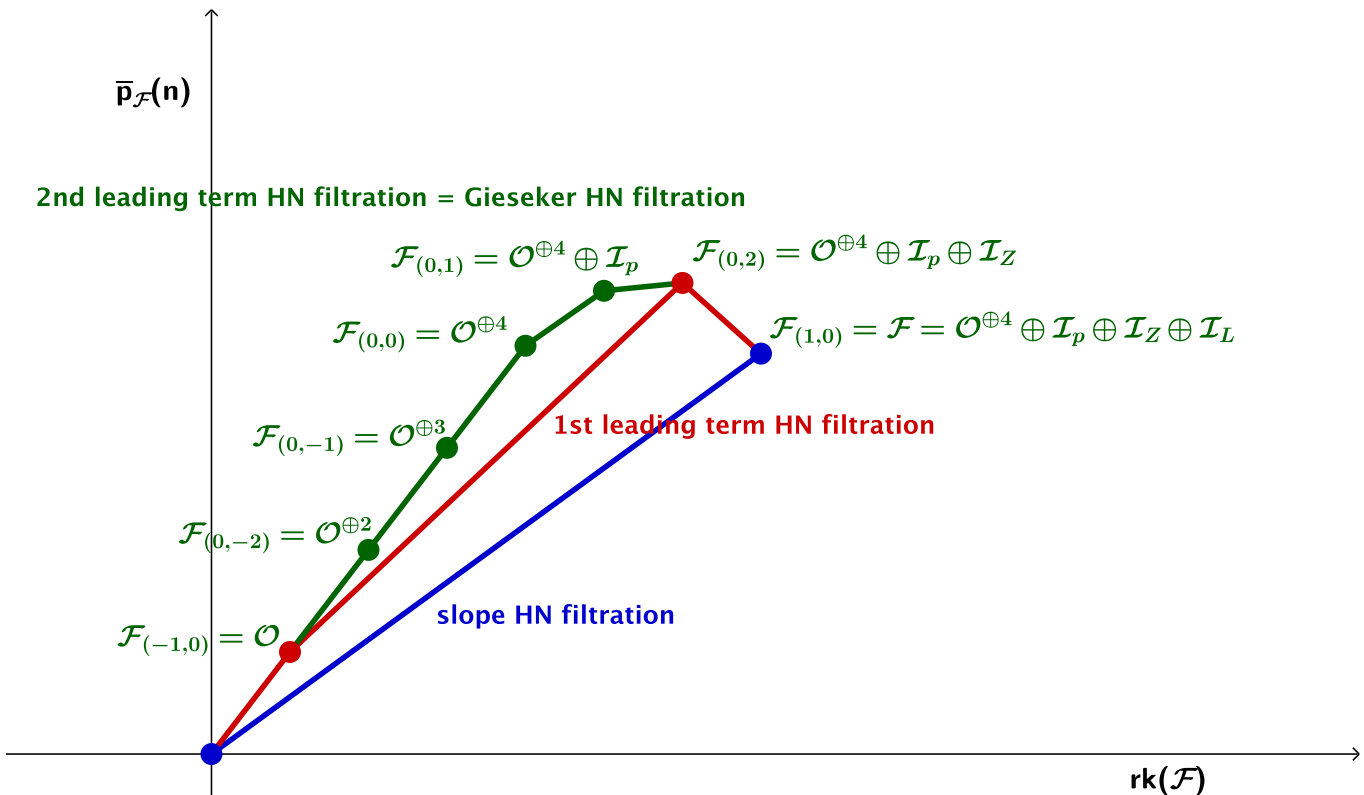
$$\begin{array}{rccc} \rho_{\lambda_2}: & \mathbb{G}_m \times (\mathbb{G}_m \times \mathbb{G}_m) & \longrightarrow & \mathrm{GL}(1)^{\times 7} \subset \mathrm{GL}(7) \\ & (a,b,c) & \mapsto & (a,b,c,1,c^{-1},b^{-1},a^{-1}) \end{array}$$

- **Associated 2^{nd} graded ρ_{λ_2}-sheaf for the 2^{nd} Levi** L_{λ_2} is isomorphic to $\mathcal{F} = \mathcal{I}_L \oplus \mathcal{I}_Z \oplus \mathcal{I}_p \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$, which is **Gieseker semiestable** (as a L_{λ_2}-sheaf) because all line bundles are $\Rightarrow$ **stop**.

Example: final Gieseker HN filtration (after 2 iterations)

Finally, the ***Gieseker Harder-Narasimhan filtration*** *is*

$$\overbrace{\mathcal{O}}^{\mathcal{F}_{(-1,0)}} \subset \overbrace{\mathcal{O}^{\oplus 2}}^{\mathcal{F}_{(0,-2)}} \subset \overbrace{\mathcal{O}^{\oplus 3}}^{\mathcal{F}_{(0,-1)}} \subset \overbrace{\mathcal{O}^{\oplus 4}}^{\mathcal{F}_{(0,0)}} \subset \overbrace{\mathcal{I}_p \oplus \mathcal{O}^{\oplus 4}}^{\mathcal{F}_{(0,1)}} \subset \overbrace{\mathcal{I}_Z \oplus \mathcal{I}_p \oplus \mathcal{O}^{\oplus 4}}^{\mathcal{F}_{(0,2)}} \subset \overbrace{\mathcal{F}}^{\mathcal{F}_{(1,0)}}$$



References


B. Anchouche, H. Azad, and I. Biswas. *Harder-Narasimhan reduction for principal bundles over a compact Kähler manifold*. Math. Ann., **323** (4): 693-712, 2002.


J. Alper, D. Halpern-Leistner, and J. Heinloth. *Existence of moduli spaces for algebraic stacks*. arXiv:1812.01128, 2019.


I. Biswas and A. Zamora. *On the Gieseker Harder-Narasimhan filtration for principal bundles*. Bull. Sci. math. **140**: 58-69, 2016.


T. L. Gómez, A. Fernández Herrero and A. Zamora. *The moduli stack of principal ρ-sheaves and Gieseker-Harder-Narasimhan filtrations*. arXiv:2107.03918, 2021.


T. L. Gómez, A. Langer, A. H. W. Schmitt, and I. Sols. *Moduli spaces for principal bundles in arbitrary characteristic*. Adv. Math., **219** (4): 1177-1245, 2008.


T. L. Gómez and I. Sols. *Moduli space of principal sheaves over projective varieties*. Ann. of Math. (2), **161** (2):1037-1092, 2005.


S. Gurjar and N. Nitsure. *Schematic Harder-Narasimhan stratification for families of principal bundles in higher dimensions*. Math. Z., **289** (3-4): 1121-1142, 2018.


D. Halpern-Leistner. *On the structure of instability in moduli theory*, arXiv:1411.0627, 2014-2022.


D. Halpern-Leistner, A. Fernandez Herrero, and T. Jones. *Moduli spaces of sheaves via affine Grassmannians*. arXiv:2107.02172, 2021.


A. Ramanathan. *Moduli of principal bundles over algebraic curves I and II*.Proc. Indian Acad. Sci. (Math. Sci.) **106** (3): 301-328 and (4): 421-449, 1996.


A. Zamora. *Harder-Narasimhan filtration for rank 2 tensors and stable coverings*. Proc. Indian Acad. Sci. (Math. Sci.) **126** (3): 305-327, 2016.