

Using reducible curves to study
Vector Bundles

Category of problems:

C general curve of genus g

what can we say about vector bundles
on C of a fixed rank and degree?

Examples: Brill Noether questions;

existence, dimension and singularities of

$W_{r,d}^k = \{E \text{ on } C \text{ of rank } r, \text{ degree } d \text{ with } k \text{ sections}\}$

Enough to find one E of rank r , degree d
with k sections and

$$H^0(E) \otimes H^0(K \otimes E^\perp) \rightarrow H^0(K \otimes E \otimes E^\perp)$$

injective

For line bundles, the Petri map is

$$H^0(L) \otimes H^0(K \otimes L^\dagger) \rightarrow H^0(K) = H^0(K \otimes L \otimes L^\dagger)$$

For vector bundles it becomes

$$H^0(E) \otimes H^0(K \otimes E^\dagger) \longrightarrow H^0(K \otimes E \otimes E^\dagger)$$

Note that $H^0(K \otimes E \otimes E^\dagger) = H^1(E \otimes E^\dagger)$

$H^1(E \otimes E^\dagger)$ parameterizes infinitesimal deformations of E . It is identified to $T_{W(r,L)}^E$ (tangent at E to moduli space of bundles)

Orthogonal to the image of the Petri map is the tangent space to the Brill-Noether locus
 $W_{r,d}^k$

More generally, find ranks of maps among spaces of sections of bundles

Example of another type of question:
Can we find E stable so that

it has a subbundle of rank $r_1 < r$ and

degree d_1 with $\frac{d_1}{r_1} < \frac{d}{r}$?

Parameterize by $H^1(\bar{E}_2 \otimes \bar{E}_1)$

More generally: given $E_1, \bar{E}_2, \dots, \bar{E}_j$ general
 $\frac{d_1}{r_1} < \frac{d_1 + \bar{d}_2}{r_1 + \bar{r}_2} < \frac{d_1 + \bar{d}_2 + \bar{d}_3}{r_1 + \bar{r}_2 + \bar{r}_3} < \dots < \frac{d_1 + \bar{d}_2 + \dots + \bar{d}_j}{r_1 + \bar{r}_2 + \dots + \bar{r}_j}$

Find E stable so that

$$0 \rightarrow E_1 \rightarrow E_2 \rightarrow \bar{E}_2 \rightarrow 0$$

$$0 \rightarrow E_2 \rightarrow E_3 \rightarrow \bar{E}_3 \rightarrow 0$$

— — —

$$0 \rightarrow E_{j-1} \rightarrow E \rightarrow \bar{E}_j \rightarrow 0$$

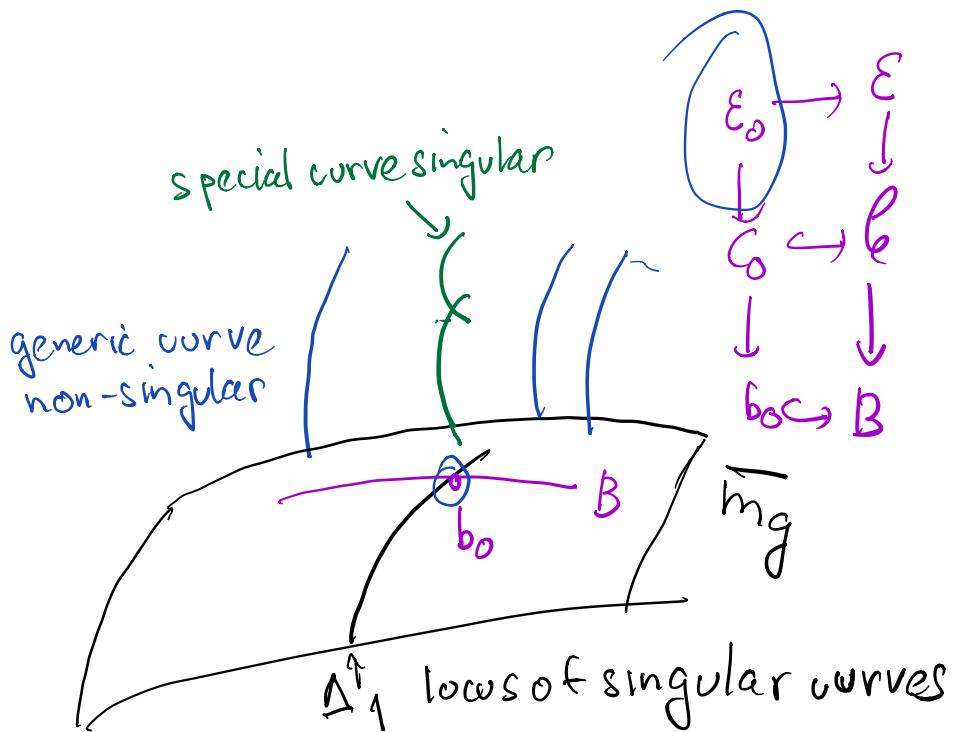
E stable

Degeneration methods :

We want to prove some result for a generic curve of genus g . The method will be useful if the property we are after is an open condition (showing it for one curve is enough).

Think of a family in \overline{M}_g with base a curve B . Assume B intersects the boundary of \overline{M}_g (has singular curves).

Get a vector bundle on the family, look at what happens on the singular curve.



Alternative category of questions

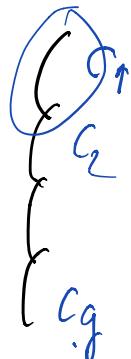
C_0 is a singular curve, what can we say about vector bundles on C_0 ?

These are very interesting questions, but we will not deal with them here except for what we need.

See poster session for many questions about singular curves.

What do we need to know about the moduli space of vector bundles on a nodal curve

- A vector bundle on C_0 is determined by choosing a vector bundle on each component and gluing at the nodes



- The notion of stability depends on the choice of a polarization.

For a non-singular curve, (semi)-stable means
 $\forall F \subset E \quad \mu(F) = \frac{\deg F}{\text{rk } F} < (\leq) \frac{\deg E}{\text{rk } E}$

For reducible curve, choose weights a_i :

$$\mu(E) = \frac{\chi(E)}{\sum a_i \text{rk } E|_{C_i}}$$

The polarization forces the degree to move on an interval of length r on each curve, once the degree on prior components is chosen).

(We can essentially ignore this condition)

Condition above essentially comes from imposing that subsheaves with support on only a few of the components do not contradict stability

We cannot ignore other destabilizing conditions

I-

- Gluing semistable vector bundles, one at least stable, the result is stable
- IF all of them are strictly semistable but we do not glue destabilizing subbundles, we are still O.K.

Additional headaches

II

- The other way around not true, there are stable bundles on \mathbb{C} with restriction to some irreducible components unstable.

- The space of vector bundles is not compact.
It can be compactified adding torsion-free sheaves (or alternatively adding vector bundles on curves with some additional rational components).

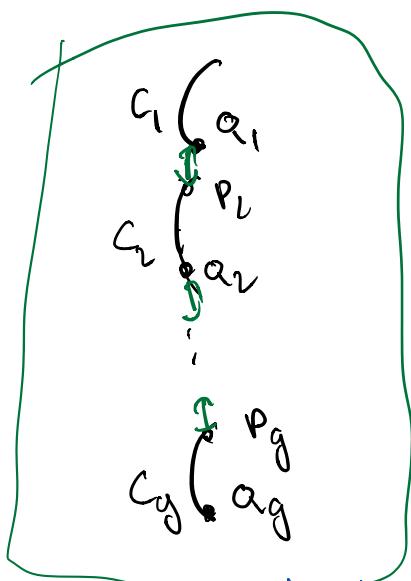
I
When proving conditions for a generic vector bundle,
I is good enough.

When proving conditions for every vector bundle
on the generic curve, we need to worry also
about II. This type of problems are much
harder

Favorite curve:
Chain of elliptic curves



Glue them together to form



Good because we understand vector bundles
(Atiyah)

For every r, d , there is a 1-dimensional family of vector bundles of rank r and degree d which is indecomposable (nota \oplus)

stable $\Leftrightarrow (r, d) = 1$

Fix r and d_1, \dots, d_g

Generic vector bundle E of rank r , degree d

multidegree (d_i) on a chain of elliptic curves.

$$r = h_i \cdot r'_i \quad d_i = h_i \cdot d'_i$$

Take: $h_i = (r, d_i)$

$$E_i = \bigoplus_{j=1}^{h_i} E_{r'_i, d'_i}^j$$

$E_{r'_i, d'_i}$ indecomposable
vector bundle of rank r'_i
degree d'_i

Each E_i depends on h_i moduli and has an h_i -dimensional family of automorphisms.

The dimension of the space of E does depend on the choice of gluings at the nodes.

The dimension of the space of vector bundles on a chain of elliptic curves is the same $r^2(g-1)+1$ as in the non-singular case.

Lange's conjecture and generalizations

Given E_1, E_2 general of ranks r_1, \bar{r}_2 and degrees d_1, \bar{d}_2 there exists a stable E such that

$$0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0 \quad \text{if} \quad \frac{d_1}{r_1} < \frac{\bar{d}_2}{\bar{r}_2}$$

Classical for $r=2$, probably in small genus.

Proved for $r=2$ using ruled surfaces

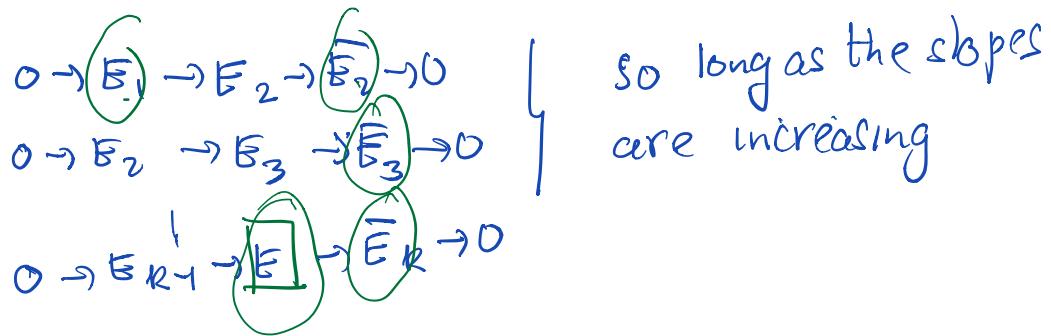
Lange asked for the generalization to arbitrary r .

Work done by many, including Ballico, Lange, Brambila-Pag. Proved in general in joint work with Russo.

Proof is essentially a parameter count: the bundles that are unstable fill a smaller subset.

Generalization

Given $E_1, \bar{E}_2, \dots, \bar{E}_K$ general there exists a stable $E = E_R$ such that



The example below gives a taste of what the proof looks like in general

Idea of proof using reducible curves

Choose $r_1=2 \quad r_2=1 \quad r_3=1 \quad g=4$

$$0 \rightarrow E_1 \rightarrow E_2 \rightarrow \bar{E}_2 \rightarrow 0$$

$$0 \rightarrow E_2 \rightarrow E_3 \rightarrow \bar{E}_3 \rightarrow 0$$

We want to construct on a curve of genus 4
 a stable vector bundle of rank 2 and degree 6
 inside a vector bundle of rank 3 and degree 13
 itself inside a vector bundle of rank 4 and
 degree 19

Recall that on an elliptic curve

$$\dim \text{Hom}(E_{r_1, d_1}, E_{r_2, d_2}) = h^0(E_{r_1, d_1}^* \otimes E_{r_2, d_2}) = \\ = r_1 d_2 - r_2 d_1$$

If $r_1 d_2 - r_2 d_1 > 0$, there exists injective
 morphism $E_{r_1, d_1} \hookrightarrow E_{r_2, d_2}$

We do this one vector bundle at a time
 on each component.

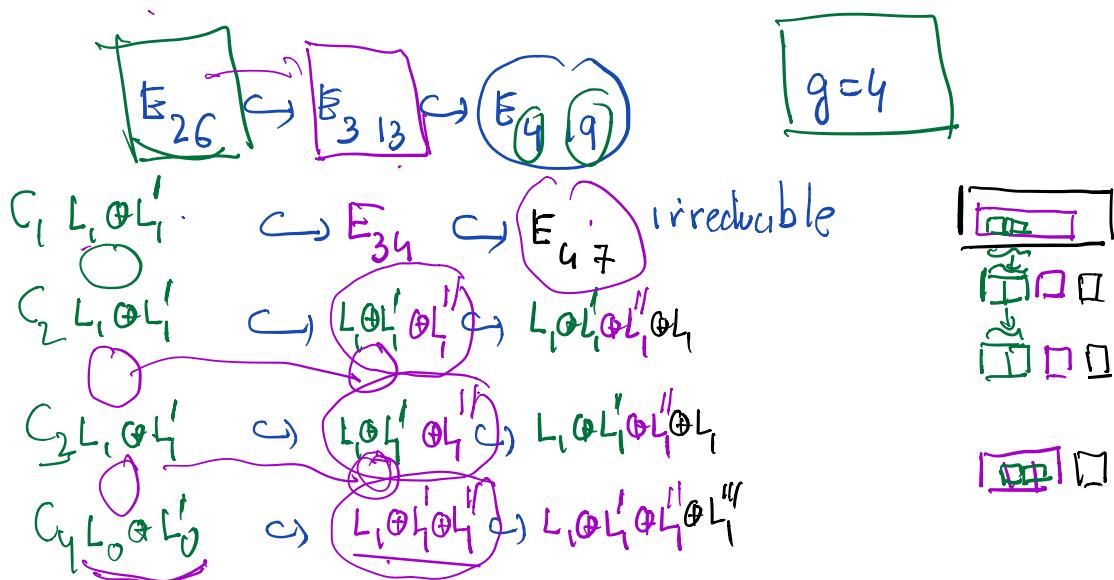
Choose the gluings to produce the

$$E_{26} \xrightarrow[\text{degree}]{} \xrightarrow[\text{rank}]{} E_{26}$$

the E_{313} (respecting the gluing of the E_{26})

and the

E_{419} (respecting the gluing of the E_{313})



First choice each restriction is semi stable
none is stable degree $2+2+2+0 = 6$

$$L_1 \oplus L_1' \\ h^0(E_{34} \otimes L_1^{-1}) = h^0(E_{34-3+1}) = h^0(E_3) = 1$$

$$h^0(E_{47} \otimes E_{34}^{-1}) = h^0(E_{12-4+7-3+4})$$

Application: Rational curves in the moduli space
of vector bundles of rank r and given determinant

Extensions $0 \rightarrow E_1 \rightarrow E \rightarrow \bar{E}_2 \rightarrow 0$
for fixed E_1, \bar{E}_2 parameterized by $H^1(\bar{E}_2^* \otimes E_1)$
There are plenty of rational curves in the
extension space

Case $r=2$, odd degree: Castravet 2004

The nice components correspond to lines in
extension spaces (two line bundles or
extensions by a skyscraper sheaf \mathcal{O}_P)

In some cases, an almost nice component
corresponding to curves of higher degree in
the extension space

General r, d (stable locus only):
(joint work with Yusuf Mustopa)

Good components still come from lines
in one step extensions.

If $h = \gcd(r, d)$ then there are h
good components of rational curves of a given
degree in $U(r, L_d)$

The multistep extensions give obstructed
components of the space of rational curves
in $U(r, L_d)$

Ranks of multiplication maps

Example 1

$$H^0(E) \otimes H^0(K \otimes E^\dagger) \rightarrow \underline{H^0(K \otimes E \otimes E^\dagger)}$$

Is the Petri map we discussed at the start

Example 2

In characteristic 0, E stalk has a 1-dimensional family of endomorphisms (homotheties) and gives direct summand

$$E \otimes E^* \cong O \oplus T_{e_0} E \quad \text{traceless endomorphisms}$$

$$H^0(K) \otimes H^0(K \otimes T_{e_0} E) \rightarrow H^0(K^2 \otimes T_{e_0} E)$$

Surjectivity has a deformation-theoretical meaning

$C \times U(r, d)$ universal bundle \mathcal{E}

$p_{2*}((p_1^* E)^\vee \otimes \mathcal{E})$ Picard bundles

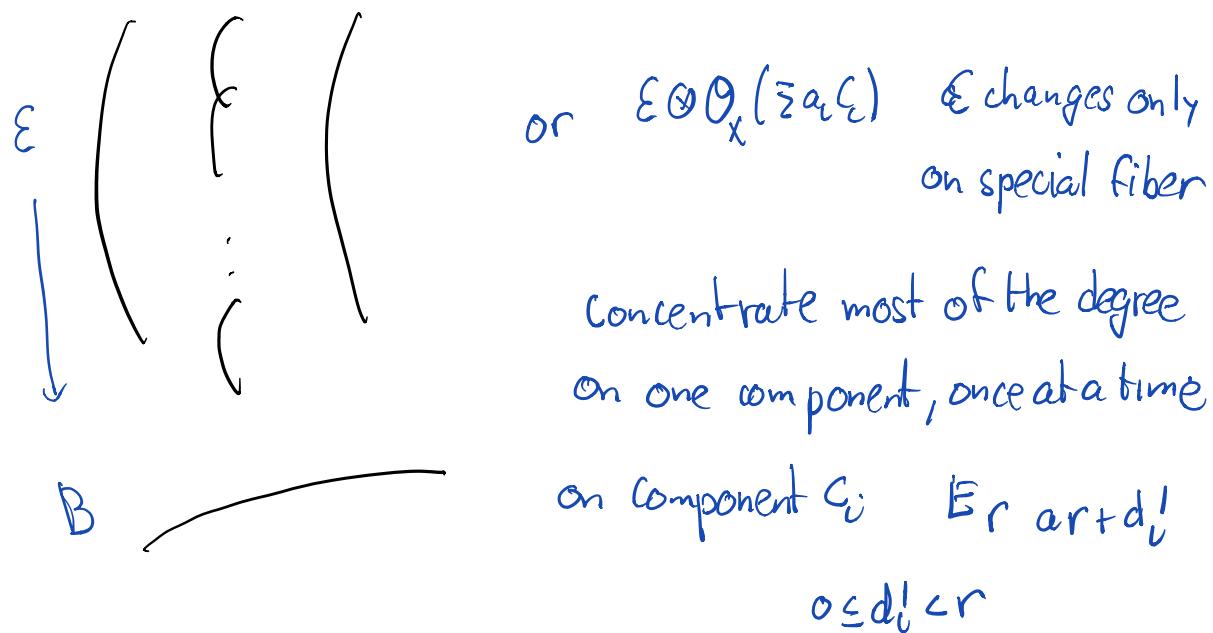
Biswas - Brambila Paez - Newstead

Moduli of Picard bundles is birational to a component of the moduli space of simple bundles on $U(r, d)$ for C general

Strategy of proof.

Use limit linear series but modify degrees so that
the support of sections is only on a few components

Idea of limit linear series



Moving from component C_i to component C_{i+1}

tensor with $\mathcal{O}_X(-\sum_{j>i} C_j)$

Sections should vanish between Q_i and P_{i+1} to order
at least a

Modified version

Instead of concentrating the degree on one component,
spread it around.

Only a few sections will be non-zero on that one
component

For example K_C

Limit linear series version		New version
c_1	$\Theta(2(g_i+1)Q_i)$	sections vanish
		$\left\{ \begin{array}{l} \deg 1 \\ \deg 2 \end{array} \right\}$
		$0 \quad \dots \quad g-1$
		$2(g_i+1) \quad 2g_i \quad \dots \quad g-2$
c_0	$\Theta(2(i-1)P_i + 2(g-i)Q_i)$	$\left\{ \begin{array}{l} \deg 2 \\ \deg 1 \end{array} \right\}$
		$i-2 \quad \dots \quad 2(g-i)$
		$2(g-i+1) \quad \dots \quad 1 \quad 0$
		\dots
		$\left\{ \begin{array}{l} \deg 1 \\ \dots \end{array} \right\}$
		$1 \quad 0$

Example in higher rank

$$E_{27}^3(C_4)$$

$$\rho(W_{27}^3(C_4)) = 2^2(4-1) + 1 - 3(3-7+2(4-1)) \\ = 13 - 3 \cdot 2 = 7$$

$$d = 7 = 2 \times 3 + 1$$

	$a=3$	$s_1 s_2 s_3$	modular	$K \otimes E^\infty$	$\bar{s}_1 \bar{s}_2 K$
C_1	$\mathcal{O}(3Q_1)^2$	$\begin{cases} 0 & 0 & 1 \\ 3 & 3 & 1 \end{cases}$	$0-4$	$\mathcal{O}(3Q_1)^2$	$\begin{cases} 3 & 3 \\ 3 & 3 \end{cases}$
C_2	$\mathcal{O}(2P_2 + Q_2) \otimes L_3$	$\begin{cases} 0 & 0 & 2 \\ 2 & 2 & 1 \end{cases}$	$4 \text{ mod 1 from gluing}$ $1-2$	$\mathcal{O}(3Q_2) \otimes L_3$	$\begin{cases} 0 & 0 \\ 3 & 2 \end{cases}$
C_3	$L \otimes L'$	$\begin{cases} 1 & 1 & 2 \\ 1 & 1 & 0 \end{cases}$	4 $2-2$	$L \otimes \bar{L}'$	$\begin{cases} 0 & 1 \\ 2 & 1 \end{cases}$
C_4	E_{27}	$\begin{cases} 2 & 2 & 3 \\ 1 & 0 & 0 \end{cases}$	3 $1-1$	\bar{E}_{25}	$\begin{cases} 1 & 2 \\ 0 & 0 \end{cases}$

$$\text{moduli: } 4+4+3-4-1+1$$

$K \otimes E \otimes E^\infty$	$s_1 \bar{s}_1$	$s_1 \bar{s}_2$	$s_2 \bar{s}_1$	$s_2 \bar{s}_2$	$s_3 \bar{s}_1$	$s_3 \bar{s}_2$
$\begin{cases} 0 \\ 6 \\ 0 \\ 5 \end{cases}$	0	0	0	0	1	1
	6	6	6	6	4	4
	0	0	0	0	2	2
	5	5				